Mixed dimensional infinite soliton trains for nonlinear Schrödinger equations

Liren Lin[*] Tai-Peng Tsai[†]

Abstract

In this note we construct mixed dimensional infinite soliton trains, which are solutions of nonlinear Schrödinger equations whose asymptotic profiles at time infinity consist of infinitely many solitons of multiple dimensions. For example infinite line-point soliton trains in 2D space, and infinite plane-line-point soliton trains in 3D space. This note extends the works of Le Coz, Li and Tsai [5, 6], where single dimensional trains are considered. In our approach, spatial L^∞ bounds for lower dimensional trains play an essential role.

Keywords: infinite soliton train, mixed dimensional, mixed train, nonlinear Schrödinger equations.

2010 Mathematics Subject Classification: 35Q55(35C08,35Q51).

1 Introduction

In this paper, we consider the nonlinear Schrödinger equation

$$i\partial_t u + \Delta u + f(u) = 0, \tag{1.1}$$

where $u = u(t,x)$ is a complex-valued function on $\mathbb{R} \times \mathbb{R}^d$, $d \geq 1$, and $f : \mathbb{C} \to \mathbb{C}$ is the nonlinearity. Our goal is to construct *mixed dimensional infinite soliton trains* (mixed trains), which are solutions of (1.1) whose asymptotic profiles at time infinity consist of infinitely many solitons of multiple dimensions.

The nonlinear Schrödinger equation (1.1) appears in various physical contexts, for example in nonlinear optics or in the modelling of Bose-Einstein condensates. Mathematically speaking, it is one of the model nonlinear dispersive PDE, along with the Korteweg-De Vries equation and the nonlinear wave equation. Its local Cauchy theory in the energy space $H^1(\mathbb{R}^d)$ is well understood (see e.g. [1] and the references cited therein). Its long time dynamics has two competing effects: First of all, if the nonlinearity is not too strong, the linear part of the equation can dominate and solutions may behave as if they were solutions to the free linear Schrödinger equation. This is the *scattering effect*. On the other hand, in some cases the nonlinear term dominates

[*]Institute of Mathematics, Academia Sinica, Taipei, Taiwan, E-mail address: b90201033@ntu.edu.tw

[†]Department of Mathematics, University of British Columbia, Vancouver BC Canada V6T 1Z2, and Center for Advanced Study in Theoretical Sciences, National Taiwan University, Taipei, Taiwan, E-mail address: ttsai@math.ubc.ca

and the solution tends to concentrate, with possible blow-up in finite time. This is the *focusing effect*. At the equilibrium between these two effects, one may encounter many different types of structures that neither scatter nor focus. The most common of these non-scattering global structures are the solitons, but there exist also dark solitons, kinks, etc. A generic conjecture for nonlinear dispersive PDE is the *Soliton Resolution Conjecture*. Roughly speaking, it says that, as can be observed in physical settings, any global solution will eventually decompose at large time into a scattering part and well separated non-scattering structures, usually a sum of solitons. Apart from integrable cases (see e.g. [13]), such conjecture is usually out of reach. Intermediate steps toward this conjecture are existence and stability results of configurations with well separated non-scattering structure, like multi-solitons, multi-kinks, infinite soliton and kink-soliton trains, etc. See [7] for a survey on these subjects. In what follows we describe results most relevant to us.

Multi-solitons are solutions of (1.1) with the asymptotic profile

$$T(t,x)=\sum_{j=1}^{N} R_j(t,x) \tag{1.2}$$

as $t\to\infty$, where $N\geq 2$ and each R_j is a soliton to be specified in (1.5). The first result of existence of multi-solitons was obtained in Zakharov and Shabat [13] in the case of the 1-d focusing cubic (i.e. $d=1$, $f(z)=|z|^2z$) nonlinear Schrödinger equation via the inverse scattering method. Indeed, in this particular case the equation is completely integrable and one can obtain multi-solitons in a rather explicit manner. Kamvissis [4] showed that it is possible to push the inverse scattering analysis forward and obtain the existence of an *infinite soliton train*, i.e. a solution u of (1.1) defined as in (1.2) but with $N=+\infty$. In fact, it is shown that, under some technical hypotheses, any solution to (1.1) with initial data in the Schwartz class will eventually decompose at large time as an infinite soliton train and a "background radiation component". There are also results for multi-dark solitons for the companion integrable Gross-Pitaevskii case, i.e. $d=1$ and $f(z)=(1-|z|^2)z$, but no known results for infinite trains.

In a non-integrable setting, the first existence result of multi-solitons was obtained by Merle in [10] as a by-product of the proof of existence of multiple blow-up points solutions for L^2-critical (1.1), i.e. $f(z)=|z|^{4/d}z$. The techniques initiated in [10] were then developed in [2, 3, 8, 9] for other nonlinearities. The idea, so called the energy method, is to choose an increasing sequence of time (t_n) with $t_n\to+\infty$ and consider the solutions (u_n) to (1.1) which solve the equation backward in time with final data $u_n(t_n)=T(t_n)$. The sequence (u_n) is an approximate sequence for a multi-soliton. To show its convergence, two arguments are at play. First, one shows that there exists a time t_0 independent of n such that u_n satisfies on $[t_0,t_n]$ the uniform estimates

$$\|(u_n-T)(t)\|_{H^1}\leq e^{-\mu\sqrt{\omega_*}v_*t}.$$

Second, we have compactness of the sequence of initial data $u_n(t_0)$, i.e. there exists u_0 so that $u_n(t_0)\to u_0$ in H^s for all $0<s<1$. See also [11, 12, 9] for stability results under restrictive hypotheses.

The energy method is very flexible and can be adapted to other situations. However, its implementation is far from being trivial when the number of solitons is infinite or when one soliton is replaced by a kink. In Le Coz, Li and Tsai [5, 6], an approach based on fixed point argument has been used to construct such structures. In this approach, the large relative speed

has been used to get smallness of the Duhamel term due to short interaction time. It is however delicate when the gradient of the error term is also measured. We will explain this approach in more details below, as we will use it to construct mixed dimensional infinite soliton trains.

We now make two assumptions on the nonlinearity f, which will be assumed throughout the paper.

Assumption (F). *$f(z) = g(|z|^2)z$, where $g \in C([0,\infty),\mathbb{R}) \cap C^2((0,\infty),\mathbb{R})$ satisfies $g(0) = 0$, and*

$$|sg'(s)| + |s^2 g''(s)| \le C_0(s^{\alpha_1/2} + s^{\alpha_2/2}) \quad (s > 0) \tag{1.3}$$

for some $C_0 > 0$, and some α_1, α_2 satisfying

$$0 < \alpha_1 \le \alpha_2 < \alpha_{\max} = \begin{cases} \infty & \text{if } d = 1, 2 \\ \frac{4}{d-2} & \text{if } d \ge 3. \end{cases}$$

If a nontrivial $\varphi \in H^1(\mathbb{R}^d, \mathbb{R})$ (bound state) and an $\omega > 0$ (frequency) satisfy

$$-\Delta\varphi + \omega\varphi = f(\varphi), \tag{1.4}$$

then for any $v \in \mathbb{R}^d$ (velocity), $x^0 \in \mathbb{R}^d$ (initial position), and $\gamma \in \mathbb{R}$ (phase),

$$R_{\varphi,\omega,v,x^0,\gamma}(t,x) := e^{i(\omega t + \frac{1}{2}v\cdot x - \frac{1}{4}|v|^2 + \gamma)}\varphi(x - x^0 - vt) \tag{1.5}$$

is a solution of (1.1), called *soliton* in this paper, in its broader meaning of *solitary wave*. The existence of solitons is a property of the nonlinearity f. To construct infinite soliton trains, we assume that there is a one parameter family of arbitrarily "small" solitons:

Assumption (T)$_d$. *For given dimension d, there are $\omega_* > 0$, $0 < a < 1$, and $D > 0$ (each depends only on d, f) such that for $0 < \omega < \omega_*$, there exist nontrivial solutions $\varphi = \varphi_\omega \in H^1(\mathbb{R}^d, \mathbb{R})$ of (1.4) satisfying*

$$|\varphi(x)| + \omega^{-\frac{1}{2}}|\nabla\varphi(x)| \le D\omega^{\frac{1}{\alpha_1}} e^{-a\omega^{1/2}|x|}, \quad \forall x \in \mathbb{R}^d. \tag{1.6}$$

Assumption (T)$_d$ is true for a large set of nonlinearities. A typical example is

$$g(s) = s^{\alpha_1/2} + cs^{\alpha_2/2}, \tag{1.7}$$

where $c \in \mathbb{R}$, and $0 < \alpha_1 < \alpha_2 < \alpha_{\max}$; see [6, Proposition 2.1] for more general nonlinearities. We shall however take it as an assumption.

In the following we discuss our problem and approach in more details.

1.1 General idea

Suppose $\{W_j(t,x)\}$ is a (finite or infinite) collection of solutions of (1.1). Intuitively, if these solutions are sufficiently separated from each other, then the nonlinear effects of their interactions should be negligible, and $\sum_j W_j$ should be close to a solution. We are interested in the

possibility that $\sum_j W_j + \eta$ is a solution for some error $\eta = \eta(t,x)$ tending to zero as $t \to \infty$. The equation of η is hence

$$\begin{cases} i\partial_t \eta + \Delta\eta + f(\sum_j W_j + \eta) - \sum_j f(W_j) = 0 \\ \eta|_{t=\infty} = 0 \quad \text{(formally)}. \end{cases}$$

By Duhamel's principle, it suffices to solve the fixed point problem

$$\eta(t) = \Phi\eta(t) := -i\int_t^\infty e^{i(t-\tau)\Delta}[G(\tau) + H(\tau)]\,d\tau, \tag{1.8}$$

where

$$\begin{aligned} G &= f(\textstyle\sum_j W_j + \eta) - f(\sum_j W_j), \\ H &= f(\textstyle\sum_j W_j) - \sum_j f(W_j). \end{aligned}$$

For a specific profile $\sum_j W_j$ in our construction, we will try to prove that Φ is a contraction mapping on a closed ball of some Banach space (see (1.9) and (1.10) below for examples), with the needed inequalities derived from the standard dispersive estimates and the Strichartz estimates. In doing so, the above decomposition of the source term into G and H will be convenient. Apparently, the control of G will come from our assumed control on η. On the other hand, the control of H is much more elaborate. It will rely on our assumption that different W_j's are sufficiently separated from each other. See the next section.

1.2 Infinite soliton trains

By an *infinite soliton train* we mean a solution of (1.1) whose asymptotic profile is of the form $T = \sum_{j\in\mathbb{N}} R_j$, where $R_j = R_{\varphi_j,\omega_j,v_j,x_j^0,\gamma_j}(t,x)$ are solitons as given by (1.5). We remark that the term "train" is only used in a suggestive sense. It well describes the one dimensional situation where all solitons travel in the same direction. In higher dimensions, the traveling directions of the constituting solitons can be rather arbitrary.

Consider $\{W_j\}$ in (1.8) to be such $\{R_j\}_{j\in\mathbb{N}}$, with $x_j^0 = 0$ for all j (see Remark 1.1 below). To give an idea of the kind of things to be proved, we cite two results (rephrased).

- [6, Theorem 1.2] For $\lambda > 0$, let $X = X_\lambda$ be the Banach space of all $\eta : [0,\infty) \times \mathbb{R}^d \to \mathbb{C}$ satisfying
$$\|\eta\|_X := \sup_{t\geq 0} e^{\lambda t}(\|\eta(t)\|_{L_x^{2+\alpha_2}} + \|\eta\|_{S(t)}) < \infty. \tag{1.9}$$
Suppose $\frac{\alpha_2}{2+\alpha_2} \leq \alpha_1$. Then, for λ large enough and under suitable conditions of $\{\omega_j\}$ and $\{v_j\}$, Φ is a contraction mapping on the closed unit ball of X_λ.

- [6, Theorem 1.6] Fix any $t_0 > 0$. For $\lambda, c > 0$, let $X = X_{\lambda,c}$ be the Banach space of all $\eta : [t_0,\infty) \times \mathbb{R}^d \to \mathbb{C}$ satisfying
$$\|\eta\|_X := \sup_{t\geq t_0}(e^{\lambda t}\|\eta\|_{S(t)} + e^{c\lambda t}\|\nabla\eta\|_{S(t)}) < \infty. \tag{1.10}$$
Suppose $0 < \alpha_1 < \frac{4}{d+2}$. Then, for suitable $c \leq 1$ and large enough λ, and under suitable conditions of $\{\omega_j\}$ and $\{v_j\}$, Φ is a contraction mapping on the closed unit ball of $X_{\lambda,c}$.

The $S(t)$ in the statements represents the Strichartz space on the time interval $[t, \infty)$ (relevant preliminaries will be given). What the "suitable conditions" are will be clear in due course. Roughly speaking, they concern the speeds of the following two limits: i) $\omega_j \to 0$, so that the Lebesgue norms of T (or also of ∇T) can be controlled by (1.6); ii) $|v_j - v_k| \to \infty$ (as $j, k \to \infty$, $j \neq k$), so that different solitons are sufficiently separated to each other.

Remark 1.1. We assume the initial positions of all the solitons to be the origin for simplicity. This is an apparent reason why some constructions (such as the second result cited above) have to be done on time intervals $[t_0, \infty)$ with positive t_0. The same situation will occur in some of our results for mixed dimensional trains.

1.3 Mixed dimensional trains

Now we consider asymptotic profiles consisting of solitons of multiple dimensions. The simplest example is

$$T_1 + T_2 := \sum_{k \in \mathbb{N}} R_{1;k}(t, x_1) + \sum_{j \in \mathbb{N}} R_{2;j}(t, x_1, x_2), \tag{1.11}$$

where $R_{1;k}$ and $R_{2;j}$ are solitons in $\mathbb{R}^1_x$ and $\mathbb{R}^2_x$ respectively. For convenience, we'll call them 1D solitons and 2D solitons. (1.11) can be visualized as the profile of a line-point soliton train in $\mathbb{R}^2_x$ (and a plane-line soliton train in $\mathbb{R}^3_x$, a space-plane soliton train in $\mathbb{R}^4_x$, and so on). Similarly, we can consider a combination of eD solitons and dD solitons for $1 \leq e < d$, or even combinations involving three or more dimensions. (It turns out that there are limited realizable combinations. See the section "Main results" below.) Solutions having such kind of profiles will be called *mixed (dimensional) trains*. In the following we take (1.11) as an example to describe the particular difficulties in constructing mixed trains.

First, our general idea encounters a problem if we only use a 2D error $\eta(t, x_1, x_2)$. To see this, note that by posing a solution of the form $T_1 + T_2 + \eta$, we get

$$H = f(T_1 + T_2) - \sum_k f(R_{1;k}) - \sum_j f(R_{2;j}).$$

Then, with x_1 being fixed, we have

$$\lim_{x_2 \to \infty} H = f(T_1(t, x_1)) - \sum_k f(R_{1;k}(t, x_1)),$$

which is nonzero in general. That is H has no space decay at infinity, and hence defies any suitable estimate (we will need L^p_x controls of H for $p \leq 2$). To resolve this problem, we will also introduce a lower dimensional error. Precisely, we will construct a solution of the form

$$T_1 + \eta_1 + T_2 + \eta,$$

where $\eta_1 = \eta_1(t, x_1)$ is such that $T_1 + \eta_1$ is an 1D train (i.e. a solution of (1.1)). In this way, by regarding $\{W_j\}$ as the sequence defined by $W_1 = T_1 + \eta_1$, and $W_{j+1} = R_{2;j}$ for $j \in \mathbb{N}$, we have

$$H = f(T_1 + \eta_1 + T_2) - f(T_1 + \eta_1) - \sum_j f(R_{2;j}),$$

which we will be able to estimate suitably.

The main difficulty in the construction is that the 1D objects $R_{1;k}$ and η_1 only allow L^∞ bounds in x_2. There are two aspects of the effect of this restriction.

1) To estimate products involving 1D objects and η (such as $\||\eta||\eta_1|^{\alpha_i}\|_{L^p_x}$), we must have $L^\infty_{x_1}$ estimates of the 1D objects, to avoid the need of dealing with "anisotropic" estimates of η. Here by an *anisotropic estimate* we mean an $L^{p_1}_{x_1}L^{p_2}_{x_2}$ estimate with $p_1 \neq p_2$. Whether such estimates are available for η is unclear (see Appendix B). Now, for $R_{1;k}$, the $L^\infty_{x_1}$ estimate is easy to obtain from (1.6). However, there is no ready result asserting an $L^\infty_{x_1}$ control of η_1. In the previous works [5, 6], the authors did not concern the possibility of constructing (single dimensional) trains with L^∞_x control of the errors. (Nevertheless, (1.10) does imply such controls by Sobolev embedding. We'll discuss it in Section 4.3.) As a consequence, we will investigate this problem before going into the mixed cases.

2) On the other hand, anisotropic estimates for $R_{2;j}$ (and $\nabla R_{2;j}$) are easy to obtain (also by (1.6)). In estimating products of them with the 1D objects, we will exploit such estimates. As will be seen, using anisotropic estimates does give us much better results.

1.4 Main results

We summarize our main results in the following.

1. From Theorem 3.7 and Theorem 3.9 (see also Corollary 3.11), there exist single dimensional trains $T + \eta$ such that
   - $\|\eta(t)\|_{L^2_x \cap L^\infty_x}$ has exponential decay in t, provided
     - $d = 1$, with $0 < \alpha_1 \le \alpha_2 < \alpha_{\max}$;
     - $d = 2, 3$, with $2(\frac{1}{2} - \frac{1}{d}) < \alpha_1 < 2$ and $\alpha_1 \le \alpha_2 < \alpha_{\max}$.
   - $\|\eta(t)\|_{H^1_x \cap W^{1,\infty}_x}$ has exponential decay in t, provided
     - $d = 1$, with $1 \le \alpha_1 < 2$ and $\alpha_1 \le \alpha_2 < \alpha_{\max}$.
2. With the above existence results of eD trains $T_e + \eta_e$ (e corresponds to the above d), Theorem 4.5 and Theorem 4.7 assert the existence of eD-dD trains $T_e + \eta_e + T_d + \eta$ such that
   - $\|\eta\|_{S(t)}$ has exponential decay in t, provided
     - $1 \le e \le 3$, $e < d \le e + 3$, with $\max(2(\frac{1}{2} - \frac{1}{e}), 0) < \alpha_1 \le \alpha_2 \le 4/d$.
   - $\|\eta\|_{S(t)} + \|\nabla\eta\|_{S(t)}$ has exponential decay in t, provided
     - $e = 1$, $d = 2$, with $1 \le \alpha_1 < 4/3$ and $\alpha_1 \le \alpha_2 < \infty$.
3. With the last result, Theorem 4.8 asserts the existence of 1D-2D-3D trains $T_1 + \eta_1 + T_2 + \eta_2 + T_3 + \eta$ such that
   - $\|\eta\|_{S(t)}$ has exponential decay in t, provided $1 \le \alpha_1 < 4/3$ and $\alpha_1 \le \alpha_2 \le 4/3$.

We give some remarks on other possible constructions, not treated in this paper.

Remark 1.2. We focus on infinite soliton trains in this paper. Our method can apparently be used to construct trains with finitely many solitons. In that case, there is no need of any assumptions on (the finite sequences) $\{\omega_j\}$ and $\{v_j\}$, as long as (1.6) is valid for all the solitons.

Remark 1.3. We may add a half kink $K(t, x_1)$ (if it exists) to one side of T_1 as in [5, 6], if all solitons (1D and higher dimensional) are positioned in the other side. If T_1 is finite, we may add half kinks to both sides. (See [5, Figure 1] for an illustration.) Notice that, in this case, it is still possible that there are infinitely many higher dimensional solitons. For example, consider the infinite 2D train profile T_2, with $v_j = (v_{j1}, v_{j2})$ being the velocities of the solitons. To combine it with T_1 having kinks on both sides, we can arrange v_{j1} to make T_2 well separated from T_1, and take $|v_{j2} - v_{\ell 2}| \to \infty$ as $j, \ell \to \infty$ ($j \neq \ell$) to make the 2D solitons to be separated from each other.

The rest of the paper is organized as follows: In Section 2, we collect some basic inequalities of the nonlinearity f. In Section 3, we construct single dimensional trains with spatial supremum control on the errors. Along the way, we give some detailed discussions as to the control of trains, which are also fundamental for mixed dimensional cases. We begin Section 4 by showing the importance of using the Strichartz estimates for constructing mixed trains. Section 4.1 gives the necessary preliminaries related to the Strichartz space. The eD-dD trains are considered in Section 4.2, and finally the 1D-2D-3D trains are constructed in Section 4.3.

2 Basic inequalities

In this section, we collect some inequalities that are simple consequences of Assumption (F). The only thing that can be said new is Proposition 2.3 (and Corollary 2.5), of which the flexibility in choosing the powers will be useful in some places. We first make the following

Convention of notation. In this paper, a constant is called *universal* if it depends only on the dimension d and the nonlinearity f, in particular C_0, α_1, α_2 in Assumption (F) and ω_*, a, D in Assumption (T)$_d$. We will use the notation $\lesssim$ in the sense that the inequality is up to a *universal multiplicative constant*. The dependence on other parameters will be given explicitly, possibly as a subscript of $\lesssim$.

Let $f_z := \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y})$ and $f_{\bar z} := \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y})$, where f is regarded as a function of $(x, y) \in \mathbb{R}^2$ by letting $f(x, y) := f(x + iy)$.

Proposition 2.1. *For $w_1, w_2 \in \mathbb{C}$, we have*

$$|f(w_1 + w_2) - f(w_1)| \lesssim \sum_{i=1,2} (|w_2||w_1|^{\alpha_i} + |w_2|^{\alpha_i+1}), \tag{2.1}$$

and

$$\begin{aligned} &|f_z(w_1 + w_2) - f_z(w_1)| + |f_{\bar z}(w_1 + w_2) - f_{\bar z}(w_1)| \\ &\qquad \lesssim \sum_{i=1,2} |w_2|^{\min(\alpha_i,1)}(|w_1| + |w_2|)^{\max(\alpha_i-1,0)}. \end{aligned} \tag{2.2}$$

See [5, Lemma 2.2] for the proofs of both inequalities. Notice that since $f(0) = 0$, (2.1) subsumes $|f(w)| \lesssim \sum_{i=1,2} |w|^{\alpha_i+1}$ for $w \in \mathbb{C}$. It's easy to check that we also have $f_z(0) = f_{\bar{z}}(0) = 0$ from Assumption (F), and (2.2) subsumes $|f_z(w)| + |f_{\bar{z}}(w)| \lesssim \sum_{i=1,2} |w|^{\alpha_i}$ for $w \in \mathbb{C}$.

For $w : \mathbb{R}^d \to \mathbb{C}$ such that the chain rule applies to $\nabla f(w(x))$ (e.g. $w \in W^{1,1}_{loc}$), it's easy to check that

$$\nabla(f(w(x))) = f_z(w(x))\nabla w(x) + f_{\bar{z}}(w(x))\overline{\nabla w(x)}. \tag{2.3}$$

We have the following corollary.

Proposition 2.2. *For $w_1, w_2 \in W^{1,1}_{loc}(\mathbb{R}^d, \mathbb{C})$, we have*

$$\begin{aligned} &|\nabla[f(w_1 + w_2) - f(w_1)]| \\ &\lesssim \sum_{i=1,2} \left\{ |w_2|^{\min(\alpha_i,1)}(|w_1| + |w_2|)^{\max(\alpha_i-1,0)}|\nabla w_1| + (|w_1| + |w_2|)^{\alpha_i}|\nabla w_2| \right\}, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} &|\nabla[f(w_1 + w_2) - f(w_1) - f(w_2)]| \\ &\lesssim \sum_{i=1,2} (|w_1| + |w_2|)^{\max(\alpha_i-1,0)}(|w_2|^{\min(\alpha_i,1)}|\nabla w_1| + |w_1|^{\min(\alpha_i,1)}|\nabla w_2|). \end{aligned} \tag{2.5}$$

Proof. Let $w = w_1 + w_2$. By (2.3),

$$\begin{aligned} |\nabla[f(w) - f(w_1)]| &= |f_z(w)\nabla w + f_{\bar{z}}(w)\overline{\nabla w} - f_z(w_1)\nabla w_1 - f_{\bar{z}}(w_1)\overline{\nabla w_1}| \\ &\le (|f_z(w) - f_z(w_1)| + |f_{\bar{z}}(w) - f_{\bar{z}}(w_1)|)|\nabla w_1| \\ &\quad + (|f_z(w)| + |f_{\bar{z}}(w)|)|\nabla w_2|, \end{aligned}$$

and (2.4) follows (2.2). For (2.5), we have

$$\begin{aligned} |\nabla[f(w) - f(w_1) - f(w_2)]| &= |f_z(w)\nabla w + f_{\bar{z}}(w)\overline{\nabla w} \\ &\quad - \sum_{j=1,2} (f_z(w_j)\nabla w_j + f_{\bar{z}}(w_j)\overline{\nabla w_j})| \\ &\le \sum_{j=1,2} (|f_z(w) - f_z(w_j)| + |f_{\bar{z}}(w) - f_{\bar{z}}(w_j)|)|\nabla w_j|. \end{aligned}$$

Let $1' = 2$ and $2' = 1$. Then (2.2) implies

$$\begin{aligned} &|\nabla[f(w) - f(w_1) - f(w_2)]| \\ &\lesssim \sum_{j=1,2} \Big\{ \sum_{i=1,2} |w_{j'}|^{\min(\alpha_i,1)}(|w_1| + |w_2|)^{\max(\alpha_i-1,0)} \Big\} |\nabla w_j| \\ &= \sum_{i=1,2} (|w_1| + |w_2|)^{\max(\alpha_i-1,0)}(|w_2|^{\min(\alpha_i,1)}|\nabla w_1| + |w_1|^{\min(\alpha_i,1)}|\nabla w_2|). \end{aligned}$$

□

Proposition 2.3. *For any $\theta_{ij}, \phi_{ij} \in [0, 1]$ ($i = 1, 2$, $j \in \mathbb{N}$), and for any absolutely convergent series $\sum_{j\in\mathbb{N}} w_j$ of complex numbers, we have*

$$|f(\sum_j w_j) - \sum_j f(w_j)| \lesssim \sum_{i=1,2} \sum_j \Big(|w_j|^{\alpha_i+\theta_{ij}} (\sum_{\ell\ne j} |w_\ell|)^{1-\theta_{ij}} + |w_j|^{1-\phi_{ij}} (\sum_{\ell\ne j} |w_\ell|)^{\alpha_i+\phi_{ij}} \Big).$$

Remark 2.4. It should be clear that we use $\sum_j$ to represent $\sum_{j\in\mathbb{N}}$, and $\sum_{\ell\neq j}$ (with j fixed) to represent $\sum_{\ell\in\mathbb{N}\setminus\{j\}}$. We'll freely use such simplified notation in this paper.

Proof. The inequality is trivial if $w_j = 0$ for all j. So assume at least one $w_j \neq 0$. Let $h_j = |w_j|/(\sum_\ell |w_\ell|)$ for each $j\in\mathbb{N}$, and let $w = \sum_j w_j$. We have

$$\begin{aligned}|f(w) - \sum_j f(w_j)| &= |\sum_j [h_j f(w) - f(w_j)]| \\ &\leq \sum_j \Big\{ h_j |f(w) - f(w_j)| + (1-h_j)|f(w_j)| \Big\}.\end{aligned}$$

By (2.1),

$$\begin{aligned}h_j|f(w) - f(w_j)| &\lesssim \frac{|w_j|}{\sum_\ell |w_\ell|} \sum_{i=1,2} \{|w-w_j||w_j|^{\alpha_i} + |w-w_j|^{\alpha_i+1}\} \\ &\leq \frac{|w_j|(\sum_{\ell\neq j}|w_\ell|)}{\sum_\ell |w_\ell|} \sum_{i=1,2} \Big\{ |w_j|^{\alpha_i} + (\sum_{\ell\neq j}|w_\ell|)^{\alpha_i}\Big\}.\end{aligned}$$

And

$$(1-h_j)|f(w_j)| \lesssim \frac{\sum_{\ell\neq j}|w_\ell|}{\sum_\ell |w_\ell|} \sum_{i=1,2} |w_j|^{\alpha_i+1} = \frac{|w_j|(\sum_{\ell\neq j}|w_\ell|)}{\sum_\ell |w_\ell|} \sum_{i=1,2} |w_j|^{\alpha_i}.$$

Thus

$$|f(w) - \sum_j f(w_j)| \lesssim \sum_{i=1,2}\sum_j \frac{|w_j|(\sum_{\ell\neq j}|w_\ell|)}{\sum_\ell |w_\ell|} \Big\{ |w_j|^{\alpha_i} + (\sum_{\ell\neq j}|w_\ell|)^{\alpha_i}\Big\}.$$

Now fix any $\theta\in[0,1]$. Notice that by Young's inequality we have

$$x + y \geq (1-\theta)^{-(1-\theta)}\theta^{-\theta}x^{1-\theta}y^\theta \geq x^{1-\theta}y^\theta, \quad \forall x, y \geq 0.$$

Thus

$$\frac{|w_j|(\sum_{\ell\neq j}|w_\ell|)}{\sum_\ell |w_\ell|} = \frac{|w_j|(\sum_{\ell\neq j}|w_\ell|)}{|w_j| + (\sum_{\ell\neq j}|w_\ell|)} \leq \frac{|w_j|(\sum_{\ell\neq j}|w_\ell|)}{|w_j|^{1-\theta}(\sum_{\ell\neq j}|w_\ell|)^\theta} = |w_j|^\theta(\sum_{\ell\neq j}|w_\ell|)^{1-\theta}.$$

This completes the proof. □

Corollary 2.5. *For any $\theta_i, \phi_i \in [0,1]$ ($i = 1,2$), and $w_1, w_2 \in \mathbb{C}$,*

$$|f(w_1+w_2) - f(w_1) - f(w_2)| \lesssim \sum_{i=1,2}\Big(|w_1|^{\alpha_i+\theta_i}|w_2|^{1-\theta_i} + |w_1|^{1-\phi_i}|w_2|^{\alpha_i+\phi_i}\Big).$$

Proof. The assertion follows by considering $w_j = 0$ for $j\geq 3$, and taking

$$\theta_{i2} = \phi_{i1} = \phi_i, \quad \phi_{i2} = \theta_{i1} = \theta_i$$

in Proposition 2.3. □

3 Single dimensional trains with L^∞_x control of errors

In this section we investigate the possibility of constructing single dimensional trains

$$T+\eta \tag{3.1}$$

such that $\|\eta(t)\|_{L^\infty_x}$ (or even $\|\eta(t)\|_{W^{1,\infty}_x}$) decays exponentially in t. Here

$$T=\sum_{j\in\mathbb{N}} R_j, \quad \text{where} \quad R_j=R_{\phi_j,\omega_j,v_j,x_j^0=0,\gamma_j}(t,x)$$

are dD solitons as given by (1.5), with $x_j^0=0$ for all j. Besides the main results (Theorem 3.7 and Theorem 3.9), many discussions in this section are also useful for next section.

By Assumption (T)$_d$,

$$\begin{aligned} |R_j(t,x)| &\le D\omega_j^{\frac{1}{\alpha_1}} e^{-a\omega_j^{1/2}|x-v_jt|}, \\ |\nabla R_j(t,x)| &\lesssim D\langle v_j\rangle \omega_j^{\frac{1}{\alpha_1}} e^{-a\omega_j^{1/2}|x-v_jt|}, \end{aligned} \tag{3.2}$$

where we used

$$|v_j|/2+\omega_j^{1/2}\lesssim\langle v_j\rangle, \quad \langle v\rangle:=(|v|^2+1)^{1/2}.$$

By the change of variable $x=\omega^{-1/2}y$, we get for $0<p\le\infty$

$$\begin{aligned} \|R_j\|_{L^p_x} &\le D_p\omega_j^{\frac{1}{\alpha_1}-\frac{d}{2p}}, \\ \|\nabla R_j\|_{L^p_x} &\lesssim D_p\langle v_j\rangle\omega_j^{\frac{1}{\alpha_1}-\frac{d}{2p}}, \end{aligned} \tag{3.3}$$

where $D_p=D\|e^{-a|y|}\|_{L^p_y}$.

Remark 3.1. The norm $\|\cdot\|_{L^p_x}$ in (3.3) is indeed $\|\cdot\|_{L^\infty(\mathbb{R},L^p(\mathbb{R}^d))}$ ($\|\cdot\|_{L^\infty_t L^p_x}$ for short). We shall however maintain the sloppy notation for simplicity. The same remark applies to $\|T\|_{L^p_x}$ and $\|\nabla T\|_{L^p_x}$, which will be considered soon. Note that as solitons do not change shapes, they can not have $L^s_tL^p_x$ bounds for any $s<\infty$.

Remark 3.2. Using the inequality $|y|\ge(|y_1|+\cdots+|y_d|)/\sqrt{d}$, we get $D_p\le D(\frac{2\sqrt{d}}{ap})^{d/p}$. Thus, for fixed $p_0>0$, $p\ge p_0$ implies $D_p\lesssim_{p_0}1$. In particular, $D_p\lesssim 1$ if p_0 is universal. There will be times we have to consider $p_0<1$.

Lemma 3.3. *For $0<p\le\infty$, and $M\ge\max(1,p^{-1})$, we have*

$$\begin{aligned} \|\sum_j|R_j|\|_{L^p_x} &\le D_p\Big(\sum_j\omega_j^{\frac{1}{M}(\frac{1}{\alpha_1}-\frac{d}{2p})}\Big)^M, \\ \|\sum_j|\nabla R_j|\|_{L^p_x} &\lesssim D_p\Big(\sum_j\langle v_j\rangle^{\frac{1}{M}}\omega_j^{\frac{1}{M}(\frac{1}{\alpha_1}-\frac{d}{2p})}\Big)^M. \end{aligned}$$

Proof. The first inequality is true by the following computation:

$$\begin{aligned}
\|\sum_j |R_j|\|_{L^p_x} &= \|(\sum_j |R_j|)^{1/M}\|^M_{L^{Mp}_x} \\
&\le \|\sum_j |R_j|^{1/M}\|^M_{L^{Mp}_x} \quad (\text{since } 1/M \le 1) \\
&\le (\sum_j \||R_j|^{1/M}\|_{L^{Mp}_x})^M \quad (\text{since } Mp \ge 1) \\
&= (\sum_j \|R_j\|^{1/M}_{L^p_x})^M.
\end{aligned}$$

Similarly, we have $\|\sum_j |\nabla R_j|\|_{L^p_x} \le (\sum_j \|\nabla R_j\|^{1/M}_{L^p_x})^M$, which implies the second inequality. $\square$

To avoid cumbersome notation, we define

$$\begin{aligned}
A_p &= A_p(\{\omega_j\}) = \Big(\sum_j \omega_j^{\min(1,p)(\frac{1}{\alpha_1}-\frac{d}{2p})}\Big)^{\max(1,p^{-1})}, \\
B_p &= B_p(\{\omega_j\},\{v_j\}) = \Big(\sum_j \langle v_j\rangle^{\min(1,p)}\omega_j^{\min(1,p)(\frac{1}{\alpha_1}-\frac{d}{2p})}\Big)^{\max(1,p^{-1})},
\end{aligned} \tag{3.4}$$

for $0 < p \le \infty$. By Lemma 3.3 (with $M = \max(1, p^{-1})$), we have

$$\|\sum_j |R_j|\|_{L^p_x} \le D_p A_p, \quad \text{and} \quad \|\sum_j |\nabla R_j|\|_{L^p_x} \lesssim D_p B_p. \tag{3.5}$$

In particular, $\|T\|_{L^p_x} \le D_p A_p$, and $\|\nabla T\|_{L^p_x} \lesssim D_p B_p$.

As discussed in the Introduction, to construct solutions of the form $T + \eta$, we consider the operator

$$\Phi\eta(t) = -i \int_t^\infty e^{i(t-\tau)\Delta}[G(\tau) + H(\tau)]\, d\tau, \tag{3.6}$$

where

$$G = f(T+\eta) - f(T), \quad \text{and} \quad H = f(T) - \sum_j f(R_j).$$

Define

$$v_* := \frac{1}{2} \inf_{j,k\in\mathbb{N}, j<k} \min(1, \omega_j^{1/2}, \omega_k^{1/2})|v_j - v_k|. \tag{3.7}$$

The following lemma gives more precise and complete estimates of H than those given in [6, Lemma 4.2, Lemma 4.4].

Lemma 3.4. *We have the following estimates for the source term H:*

(H0) *Fix any $r_0 > 0$. For $r > s > r_0$ and $t \ge 0$,*

$$\|H(t)\|_{L^r_x} \lesssim_{r_0} (\sum_{i=1,2} A^{\alpha_i+1}_{(\alpha_i+1)s})^{s/r} (\sum_{i=1,2} A^{\alpha_i+1}_\infty)^{1-s/r} e^{-a(1-s/r)v_* t}.$$

(H1) *Fix any* $r_1 > 0$. *For* $r > s > r_1$ *and* $t \geq 0$,

$$\|\nabla H(t)\|_{L^r_x} \lesssim_{r_1} \Big(\sum_{i=1,2} A^{\alpha_i}_{\alpha_i q} B_p\Big)^{s/r} \Big(\sum_{i=1,2} A^{\alpha_i}_{\infty} B_\infty\Big)^{1-s/r} e^{-a\min(\alpha_1,1)(1-s/r)v_* t},$$

where p, q *are arbitrary numbers in* $(0,\infty]$ *satisfying* $\frac{1}{q} + \frac{1}{p} = \frac{1}{s}$.

Remark 3.5. The inequalities are indeed true for all $r > s > 0$, only that the multiplicative constants will then depend on s. For the upper bounds given in (H0) and (H1) to be under desirable control, there are actually natural choices of r_0 and r_1 that are universal (depending only on d, α_1, α_2). We'll discuss this point right after the proof.

Proof of Lemma 3.4. Each assertion is proved by the same strategy as in [5, 6]: Prove the exponential decay in t of the L^∞_x norm, by singling out the soliton "nearest" to a fixed (x,t). And prove the boundedness of the L^s_x norm independent of t. Then the L^r_x estimate follows by interpolation.

PROOF OF (H0). For fixed t, x, let $m = m(t,x) \in \mathbb{N}$ be such that

$$|x - v_m t| = \min_{j\in\mathbb{N}} |x - v_j t|.$$

Then for $j \neq m$,

$$|x - v_j t| = |x - v_m t + (v_m - v_j)t| \geq |v_j - v_m|t - |x - v_m t| \geq |v_j - v_m|t - |x - v_j t|,$$

and hence

$$|x - v_j t| \geq \frac{1}{2}|v_j - v_m|t. \tag{3.8}$$

By (2.1),

$$\begin{aligned}
|H| &\leq |f(T) - f(R_m)| + \sum_{j\neq m} |f(R_j)| \\
&\lesssim \sum_{i=1,2} \Big\{ |T - R_m|(|R_m| + |T - R_m|)^{\alpha_i} + \sum_{j\neq m} |R_j|^{\alpha_i+1} \Big\} \\
&\leq \sum_{i=1,2} \Big\{ \Big(\sum_{j\neq m} |R_j|\Big)\Big(\sum_{j} |R_j|\Big)^{\alpha_i} + \Big(\sum_{j\neq m} |R_j|\Big)^{\alpha_i+1} \Big\}.
\end{aligned}$$

Thus, by (3.2) and the definition of v_*,

$$\begin{aligned}
|H| &\lesssim \sum_{i=1,2} \Big\{ \Big(\sum_{j\neq m} \omega_j^{\frac{1}{\alpha_1}} e^{-av_* t}\Big)\Big(\sum_{j} \omega_j^{\frac{1}{\alpha_1}}\Big)^{\alpha_i} + \Big(\sum_{j\neq m} \omega_j^{\frac{1}{\alpha_1}} e^{-av_* t}\Big)^{\alpha_i+1} \Big\} \\
&\lesssim \sum_{i=1,2} \Big\{ \Big(\sum_{j\neq m} \omega_j^{\frac{1}{\alpha_1}}\Big)\Big(\sum_{j} \omega_j^{\frac{1}{\alpha_1}}\Big)^{\alpha_i} + \Big(\sum_{j\neq m} \omega_j^{\frac{1}{\alpha_1}}\Big)^{\alpha_i+1} \Big\} e^{-av_* t} \quad (t \geq 0) \\
&\lesssim \Big(\sum_{i=1,2} A_\infty^{\alpha_i+1}\Big) e^{-av_* t}.
\end{aligned}$$

Now that the upper bound is independent of x and m, we get

$$\|H(t)\|_{L_x^\infty} \lesssim (\sum_{i=1,2} A_\infty^{\alpha_i+1})e^{-av_*t}. \tag{3.9}$$

Next, we try to bound $\|H\|_{L_x^s}$ for finite $s > r_0 > 0$. By Proposition 2.3, in particular its flexibility of choosing θ_{ij} and ϕ_{ij},

$$\begin{aligned}
|H| &\lesssim \sum_{i=1,2}\sum_j \Big\{|R_j|^{\max(\alpha_i,1)}(\sum_{\ell\neq j}|R_\ell|)^{\min(\alpha_i,1)} + |R_j|(\sum_{\ell\neq j}|R_\ell|)^{\alpha_i}\Big\} \\
&\leq \sum_{i=1,2}\sum_j \Big\{|R_j|^{\max(\alpha_i,1)}(\sum_{\ell}|R_\ell|)^{\min(\alpha_i,1)} + |R_j|(\sum_{\ell}|R_\ell|)^{\alpha_i}\Big\} \\
&\leq \sum_{i=1,2}\Big\{(\sum_j |R_j|^{\max(\alpha_i,1)})(\sum_{\ell}|R_\ell|)^{\min(\alpha_i,1)} + (\sum_j|R_j|)(\sum_{\ell}|R_\ell|)^{\alpha_i}\Big\}.
\end{aligned}$$

Since $\sum_j |R_j|^{\max(\alpha_i,1)} \leq (\sum_j |R_j|)^{\max(\alpha_i,1)}$ due to $\max(\alpha_i,1)\geq 1$, we get

$$|H| \lesssim \sum_{i=1,2}(\sum_j |R_j|)^{\alpha_i+1}.$$

Thus, for $s > r_0$, by (3.5) (and Remark 3.2)

$$\|H\|_{L_x^s} \lesssim \sum_{i=1,2}\|\sum_j |R_j|\|^{\alpha_i+1}_{L_x^{(\alpha_i+1)s}} \lesssim_{r_0} \sum_{i=1,2} A^{\alpha_i+1}_{(\alpha_i+1)s}. \tag{3.10}$$

By (3.9) and (3.10), for $r > s > r_0$, we have

$$\|H\|_{L_x^r} \leq \|H\|_{L_x^s}^{s/r}\|H\|_{L_x^\infty}^{1-s/r} \lesssim_{r_0} (\sum_{i=1,2} A^{\alpha_i+1}_{(\alpha_i+1)s})^{s/r}(\sum_{i=1,2} A_\infty^{\alpha_i+1})^{1-s/r}e^{-a(1-s/r)v_*t}.$$

PROOF OF (H1). By (2.3) and (2.2),

$$\begin{aligned}
|\nabla H| &\leq \sum_j (|f_z(T)-f_z(R_j)| + |f_{\bar z}(T) - f_{\bar z}(R_j)|)|\nabla R_j| \\
&\lesssim \sum_{i=1,2}\sum_j (|T-R_j|)^{\min(\alpha_i,1)}(|T-R_j| + |R_j|)^{\max(\alpha_i-1,0)}|\nabla R_j| \\
&\leq \sum_{i=1,2}\sum_j (\sum_{\ell\neq j}|R_\ell|)^{\min(\alpha_i,1)}(\sum_\ell |R_\ell|)^{\max(\alpha_i-1,0)}|\nabla R_j| \\
&\lesssim \sum_{i=1,2} A_\infty^{\max(\alpha_i-1,0)}E_i,
\end{aligned} \tag{3.11}$$

where

$$E_i := \sum_j (\sum_{\ell\neq j}|R_\ell|)^{\min(\alpha_i,1)}|\nabla R_j|.$$

Let $m = m(t,x) \in \mathbb{N}$ be as above. By (3.2) and (3.8),

$$\begin{aligned} E_i &\lesssim \sum_{j\neq m}(\sum_{\ell\neq j}|R_\ell|)^{\min(\alpha_i,1)}|\nabla R_j| + (\sum_{\ell\neq m}|R_\ell|)^{\min(\alpha_i,1)}|\nabla R_m| \\ &\lesssim \sum_{j\neq m} A_\infty^{\min(\alpha_i,1)}|\nabla R_j| + (\sum_{\ell\neq m}|R_\ell|)^{\min(\alpha_i,1)}B_\infty \\ &\lesssim A_\infty^{\min(\alpha_i,1)}B_\infty e^{-av_*t} + B_\infty(A_\infty e^{-av_*t})^{\min(\alpha_i,1)} \\ &\le (2A_\infty^{\min(\alpha_i,1)}B_\infty)e^{-a\min(\alpha_i,1)v_*t}. \quad (t\ge 0) \end{aligned}$$

Thus

$$\|\nabla H(t)\|_{L_x^\infty} \lesssim (\sum_{i=1,2} A_\infty^{\alpha_i}B_\infty)e^{-a\min(\alpha_1,1)v_*t}. \tag{3.12}$$

On the other hand, from (3.11),

$$\begin{aligned} |\nabla H| &\lesssim \sum_{i=1,2}\sum_j(\sum_\ell|R_\ell|)^{\min(\alpha_i,1)}(\sum_\ell|R_\ell|)^{\max(\alpha_i-1,0)}|\nabla R_j| \\ &= \sum_{i=1,2}(\sum_\ell|R_\ell|)^{\alpha_i}(\sum_j|\nabla R_j|). \end{aligned}$$

Hence, for $s > r_1$,

$$\|\nabla H\|_{L_x^s} \lesssim \sum_{i=1,2}\|(\sum_\ell|R_\ell|)^{\alpha_i}\|_{L_x^q}\|\sum_j|\nabla R_j|\|_{L^p} \lesssim_{r_1} \sum_{i=1,2} A_{\alpha_i q}^{\alpha_i}B_p, \tag{3.13}$$

where p,q are any numbers in $(0,\infty]$ satisfying $\frac{1}{q}+\frac{1}{p}=\frac{1}{s}$. By (3.12) and (3.13), for $r > s > r_1$, we have

$$\|\nabla H(t)\|_{L_x^r} \lesssim_{r_1} (\sum_{i=1,2}A_{\alpha_i q}^{\alpha_i}B_p)^{s/r}(\sum_{i=1,2}A_\infty^{\alpha_i}B_\infty)^{1-s/r}e^{-a\min(\alpha_1,1)(1-s/r)v_*t}. \qquad \square$$

Now we explain how the values of A_p, B_p (here p is regarded as a parameter) and v_* should be controlled, by adjusting $\{\omega_j\}$ and $\{v_j\}$ of the profile T. As is mentioned, we need $\omega_j \to 0$ and $|v_j - v_k| \to \infty$. Precisely, *we will need the flexibility of making v_* as large as we like, and at the same time controlling the sizes of A_p and B_p.* As to this purpose, the first obvious observation is that $A_p < \infty$ can be true if and only if $\frac{1}{\alpha_1} - \frac{d}{2p} > 0$, i.e. $p > \frac{d\alpha_1}{2}$. Next, a little thought shows that $v_* > 0$ and $B_p < \infty$ can hold simultaneously only if $\frac{1}{\alpha_1} - \frac{d}{2p} > \frac{1}{2}$, which is equivalent to $\alpha_1 < 2$ and $p > \frac{d\alpha_1}{2-\alpha_1}$. It turns out that these minimum requirements are sufficient. We give the relevant facts in the next lemma. For convenience, we define

$$\mathcal{C}_A = (\frac{d\alpha_1}{2},\infty]; \quad \mathcal{C}_B = (\frac{d\alpha_1}{2-\alpha_1},\infty] \quad (\text{if } \alpha_1 < 2). \tag{3.14}$$

Lemma 3.6.

(a) *For $\infty \ge q > p \in \mathcal{C}_A$, we have $A_q < \max(1,\omega_*)^{\frac{1}{\alpha_1}}A_p$ whenever $A_p < \infty$. If $\alpha_1 < 2$ and $\infty \ge q > p \in \mathcal{C}_B$, we have $B_q < \max(1,\omega_*)^{\frac{1}{\alpha_1}}B_p$ whenever $B_p < \infty$.*

(b) *Suppose $q \in \mathcal{C}_A$, then for any constants $c, \Lambda > 0$, there exist $\{\omega_j\}$ and $\{v_j\}$ such that $A_q \le c$, and $v_* \ge \Lambda$. If moreover $\alpha_1 < 2$ and $p \in \mathcal{C}_B$, then $\{\omega_j\}$ and $\{v_j\}$ can be chosen so that $B_p \le c$ is also satisfied.*

The proofs of these facts are elementary and are given in Appendix A. Briefly, (a) says $A_q \lesssim A_p$ and $B_q \lesssim B_p$ for $q \ge p$. As a consequence, when there are several A_p or B_p to be controlled, it suffices to control those having smaller p. And (b) is exactly the desired control. (a) and (b) will be fundamental for the effectiveness of our estimates of G and H.

For the construction of soliton trains in this section, the needed estimates will be derived from the dispersive inequality: If $p \in [2, \infty]$ and $t \ne 0$,

$$\|e^{it\Delta}u\|_{L^p(\mathbb{R}^d)} \le (4\pi|t|)^{-d(\frac{1}{2}-\frac{1}{p})}\|u\|_{L^{p'}(\mathbb{R}^d)} \quad \forall\, u \in L^{p'}(\mathbb{R}^d). \tag{3.15}$$

We now give our first main result.

Theorem 3.7. *Let $d = 1$, and f satisfy Assumptions (F) and (T)$_d$. Suppose moreover $\alpha_1 \ge 1$. Then for any finite $\rho > 0$, there is a constant $\lambda_0 > 0$ such that the following holds: For $\lambda_0 \le \lambda < \infty$, there exist solutions of* (1.1) *of the form* (3.1) *for $t \ge 0$, with*

$$\sup_{t\ge 0} e^{\lambda t}\|\eta(t)\|_{L^2_x \cap L^\infty_x} \le \rho. \tag{3.16}$$

Proof. For $0 < \lambda < \infty$, let $X = X_\lambda$ be the Banach space of all $\eta : [0,\infty) \times \mathbb{R}^1 \to \mathbb{C}$ with norm $\|\eta\|_X$ defined by the left-hand side of (3.16). By interpolation, we have

$$\|\eta(t)\|_{L^p_x} \le \|\eta\|_X e^{-\lambda t} \quad \forall\, p \in [2,\infty], \quad \forall t \ge 0. \tag{3.17}$$

Given $\rho \in (0,\infty)$, we will prove that, for sufficiently large λ, there are $\{\omega_j\}, \{v_j\}$ such that Φ (defined in (3.6)) is a contraction mapping on the closed ball $\{\eta \in X : \|\eta\|_X \le \rho\}$.

First, we give estimates for Φ to be a self-mapping. Given $\eta \in X$ with $\|\eta\|_X \le \rho$. For $p \in [2,\infty]$, the dispersive inequality (3.15) implies

$$\|\Phi\eta(t)\|_{L^p_x} \lesssim \int_t^\infty |t-\tau|^{-(\frac{1}{2}-\frac{1}{p})}(\|G(\tau)\|_{L^{p'}_x} + \|H(\tau)\|_{L^{p'}_x})\, d\tau.$$

To estimate $\|\Phi\eta\|_X$, we have to estimate $\|G(\tau)\|_{L^2_x}$, $\|G(\tau)\|_{L^1_x}$, $\|H(\tau)\|_{L^2_x}$ and $\|H(\tau)\|_{L^1_x}$.

By (2.1),

$$|G| = |f(T+\eta) - f(T)| \lesssim \sum_{i=1,2} \left\{|\eta||T|^{\alpha_i} + |\eta|^{\alpha_i+1}\right\}.$$

For the first term, we have

$$\||\eta||T|^{\alpha_i}(\tau)\|_{L^2_x} \le \|\eta(\tau)\|_{L^2_x}\|T\|^{\alpha_i}_{L^\infty_x} \lesssim \rho A^{\alpha_i}_\infty e^{-\lambda\tau}, \tag{3.18}$$

$$\||\eta||T|^{\alpha_i}(\tau)\|_{L^1_x} \le \|\eta(\tau)\|_{L^2_x}\|T\|^{\alpha_i}_{L^{2\alpha_i}_x} \lesssim \rho A^{\alpha_i}_{2\alpha_i} e^{-\lambda\tau}, \tag{3.19}$$

where notice that $2\alpha_1 \in \mathcal{C}_A$. For the second term, by (3.17),

$$\||\eta|^{\alpha_i+1}(\tau)\|_{L^2_x} = \|\eta(\tau)\|^{\alpha_i+1}_{L^{2(\alpha_i+1)}_x} \le \rho^{\alpha_i+1}e^{-(\alpha_i+1)\lambda\tau} \le \rho^{\alpha_i+1}e^{-\lambda\tau}, \tag{3.20}$$

$$\||\eta|^{\alpha_i+1}(\tau)\|_{L^1_x} = \|\eta(\tau)\|^{\alpha_i+1}_{L^{\alpha_i+1}_x} \le \rho^{\alpha_i+1}e^{-(\alpha_i+1)\lambda\tau} \le \rho^{\alpha_i+1}e^{-\lambda\tau}, \tag{3.21}$$

where in (3.21) we use the assumption $\alpha_1 \geq 1$.

For H, taking $r = 2$ and $s = 1$ in Lemma 3.4 (H0), we get

$$\|H(\tau)\|_{L_x^2} \lesssim (\sum_{i=1,2} A_{\alpha_i+1}^{\alpha_i+1})^{1/2} (\sum_{i=1,2} A_\infty^{\alpha_i+1})^{1/2} e^{-\frac{a}{2} v_* \tau}, \tag{3.22}$$

with $\alpha_1 + 1 \in \mathcal{C}_A$. Taking $r = 1$ and $s = 1/2$, we get

$$\|H(\tau)\|_{L_x^1} \lesssim (\sum_{i=1,2} A_{(\alpha_i+1)/2}^{\alpha_i+1})^{1/2} (\sum_{i=1,2} A_\infty^{\alpha_i+1})^{1/2} e^{-\frac{a}{2} v_* \tau}, \tag{3.23}$$

with $(\alpha_1 + 1)/2 \in \mathcal{C}_A$.

Now suppose

$$\frac{a}{2} v_* \geq \lambda. \tag{3.24}$$

Then from (3.18), (3.20) and (3.22), we get

$$\|\Phi\eta(t)\|_{L_x^2} \lesssim \int_t^\infty E_1 e^{-\lambda\tau}\, d\tau = E_1 \lambda^{-1} e^{-\lambda t}, \tag{3.25}$$

where

$$E_1 = \sum_{i=1,2} (\rho A_\infty^{\alpha_i} + \rho^{\alpha_i+1}) + (\sum_{i=1,2} A_{\alpha_i+1}^{\alpha_i+1})^{1/2} (\sum_{i=1,2} A_\infty^{\alpha_i+1})^{1/2}.$$

From (3.19), (3.21) and (3.23), we get

$$\|\Phi\eta(t)\|_{L_x^\infty} \lesssim \int_t^\infty |t-\tau|^{-\frac{1}{2}} E_2 e^{-\lambda\tau}\, d\tau = E_2 \Gamma(1/2) \lambda^{-1/2} e^{-\lambda t}, \tag{3.26}$$

where

$$E_2 = \sum_{i=1,2} (\rho A_{2\alpha_i}^{\alpha_i} + \rho^{\alpha_i+1}) + (\sum_{i=1,2} A_{(\alpha_i+1)/2}^{\alpha_i+1})^{1/2} (\sum_{i=1,2} A_\infty^{\alpha_i+1})^{1/2},$$

and $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$ is the Gamma function.

Next, we give estimates for contractivity. Given $\eta_1, \eta_2 \in X$, $\|\eta_1\|_X, \|\eta_2\|_X \leq \rho$. We have

$$\Phi\eta_1 - \Phi\eta_2 = -i \int_t^\infty e^{i(t-\tau)\Delta} [f(T+\eta_1) - f(T+\eta_2)](\tau)\, d\tau.$$

Hence, for $p \in [2, \infty]$,

$$\|(\Phi\eta_1 - \Phi\eta_2)(t)\|_{L_x^p} \lesssim \int_t^\infty |t-\tau|^{-(\frac{1}{2}-\frac{1}{p})} \|[f(T+\eta_1) - f(T+\eta_2)](\tau)\|_{L_x^{p'}}\, d\tau.$$

By (2.1),

$$\begin{aligned}
|f(T+\eta_1) - f(T+\eta_2)| &\lesssim \sum_{i=1,2} \left\{ |\eta_1 - \eta_2| |T+\eta_2|^{\alpha_i} + |\eta_1 - \eta_2|^{\alpha_i+1} \right\} \\
&\lesssim \sum_{i=1,2} \{ |\eta_1 - \eta_2| |T|^{\alpha_i} + |\eta_1 - \eta_2| (|\eta_1| + |\eta_2|)^{\alpha_i} \}.
\end{aligned}$$

By (3.17),

$$\begin{aligned}\||\eta_1-\eta_2||T|^{\alpha_i}(\tau)\|_{L^2_x} &\le \|(\eta_1-\eta_2)(\tau)\|_{L^2_x}\|T\|^{\alpha_i}_{L^\infty_x}\\ &\lesssim A^{\alpha_i}_\infty\|\eta_1-\eta_2\|_X e^{-\lambda\tau},\end{aligned}$$

$$\begin{aligned}\||\eta_1-\eta_2||T|^{\alpha_i}(\tau)\|_{L^1_x} &\le \|(\eta_1-\eta_2)(\tau)\|_{L^2_x}\|T(\tau)\|^{\alpha_i}_{L^{2\alpha_i}_x}\\ &\lesssim A^{\alpha_i}_{2\alpha_i}\|\eta_1-\eta_2\|_X e^{-\lambda\tau},\end{aligned}$$

$$\begin{aligned}\||\eta_1-\eta_2|(|\eta_1|+|\eta_2|)^{\alpha_i}(\tau)\|_{L^2_x} &\le \|(\eta_1-\eta_2)(\tau)\|_{L^2_x}\|(|\eta_1|+|\eta_2|)(\tau)\|^{\alpha_i}_{L^\infty_x}\\ &\le (2\rho)^{\alpha_i}\|\eta_1-\eta_2\|_X e^{-\lambda\tau},\end{aligned}$$

$$\begin{aligned}\||\eta_1-\eta_2|(|\eta_1|+|\eta_2|)^{\alpha_i}(\tau)\|_{L^1_x} &\le \|(\eta_1-\eta_2)(\tau)\|_{L^2_x}\|(|\eta_1|+|\eta_2|)(\tau)\|^{\alpha_i}_{L^{2\alpha_i}_x}\\ &\le (2\rho)^{\alpha_i}\|\eta_1-\eta_2\|_X e^{-\lambda\tau}.\end{aligned}$$

Hence

$$\|(\Phi\eta_1-\Phi\eta_2)(t)\|_{L^2_x} \lesssim E_3\lambda^{-1}\|\eta_1-\eta_2\|_X e^{-\lambda t}, \tag{3.27}$$

where $E_3=\sum_{i=1,2}(A^{\alpha_i}_\infty+(2\rho)^{\alpha_i})$. And

$$\|(\Phi\eta_1-\Phi\eta_2)(t)\|_{L^\infty_x} \lesssim E_4\Gamma(1/2)\lambda^{-1/2}\|\eta_1-\eta_2\|_X e^{-\lambda t}, \tag{3.28}$$

where $E_4=\sum_{i=1,2}(A^{\alpha_i}_{2\alpha_i}+(2\rho)^{\alpha_i})$.

Now for any $\lambda>0$, Lemma 3.6 ensures that we can choose $\{\omega_j\}$ and $\{v_j\}$ (depending on λ) such that v_* satisfies (3.24), with all "A_p" being no larger than any preassigned number, say $A_p\le 1$. In particular, we see there is a constant $E=E(\alpha_1,\alpha_2,\rho)>0$ such that $E_\ell\le E$ for $\ell=1,2,3,4$, given in (3.25), (3.26), (3.27), and (3.28). Thus, also from these inequalities, if λ is large enough (i.e. $\lambda\in[\lambda_0,\infty)$ for some large enough λ_0), such choice of $\{\omega_j\},\{v_j\}$ gives

$$\begin{aligned}\|\Phi\eta(t)\|_{L^2_x\cap L^\infty_x} &\le \rho e^{-\lambda t}\\ \|(\Phi\eta_1-\Phi\eta_2)(t)\|_{L^2_x\cap L^\infty_x} &\le \frac{1}{2}\|\eta_1-\eta_2\|_X e^{-\lambda t}.\end{aligned}$$

Hence Φ is a contraction mapping on the closed ball of X with radius ρ. □

Remark 3.8. By the contraction mapping principle, for a fixed profile T such that Φ is a contraction, the error η is unique within the class we try to find it.

Before giving the next theorem, we make some comments on the choices of $\{\omega_j\}$ and $\{v_j\}$. As gradient estimate of η is not involved in the previous proof, B_p does not occur, and the last part of the proof can be replaced by the following: 1) First choose $\{\omega_j\}$ so that the coefficients $E_1\sim E_4$ are finite (equivalently, all "A_p" are finite), then 2) choose $\lambda\ge\lambda_0$ sufficiently large so that (3.25) – (3.28) imply that Φ is a contraction mapping. And hence 3) the construction is done for any $\{v_j\}$ such that (3.24) is satisfied.

It's then easy to see what choices of $\{\omega_j\}, \{v_j\}$ are allowable. For example, since $A_{(\alpha_1+1)/2}$ is the A_p with smallest p to be controlled in the proof, the construction is possible if and only if $\{\omega_j\}$ is such that $A_{(\alpha_1+1)/2} < \infty$, i.e.

$$\sum_j \omega_j^{\frac{1}{\alpha_1}-\frac{1}{\alpha_1+1}} < \infty. \tag{3.29}$$

However, when there is B_p, the Step 3) of choosing $\{v_j\}$ will also influence the coefficients considered in Step 1). For later considerations, we have given a proof that works even when B_p is present: For every λ, Lemma 3.6 ensures that we can choose $\{\omega_j\}$ and $\{v_j\}$ so that v_* is large enough and all A_p, B_p are small. For large enough λ, Φ is hence a contraction mapping for such $\{\omega_j\}, \{v_j\}$. Moreover, giving precise conditions as (3.29), though possible, would be rather cumbersome. We shall hence satisfy ourselves with such vague statement as Theorem 3.7. Suffice it to say that, once a construction is done with some choice of $\{\omega_j\}$ and $\{v_j\}$, it is done with all other choices making the present A_p and B_p smaller and the v_* larger. One easy way to obtain such "better" choices is by rescaling, i.e. by considering $\{\kappa\omega_j\}$ and $\{\nu v_k\}$ for suitable positive constants κ, ν. The argument is routine and we omit the details.

We now turn to our next main result. First, notice that the proof of Theorem 3.7 fails for $d \geq 2$, since the dispersive inequality gives

$$\|\Phi\eta(t)\|_{L^\infty_x} \lesssim \int_t^\infty |t-\tau|^{-\frac{d}{2}}(\|G(\tau)\|_{L^1_x} + \|H(\tau)\|_{L^1_x})\, d\tau,$$

where the singularity at $\tau = t$ is not integrable. As a consequence, we consider the following alternative way: Construct trains $T + \eta$ having $\|\eta(t)\|_{L^2_x}$ and $\|\nabla\eta(t)\|_{L^r_x}$ controls for some $r > d$. Then the $\|\eta(t)\|_{L^\infty_x}$ control follows from Sobolev embedding (Gagliardo-Nirenberg's inequality). It turns out that we still need $d \leq 3$. Moreover, due to some technical benefits, we also assume the $\|\nabla\eta(t)\|_{L^2_x}$ control in our construction (see Remark 3.10 after the proof).

Theorem 3.9. *Let $d \leq 3$, and f satisfy Assumptions (F) and (T)$_d$. Suppose $2(\frac{1}{2}-\frac{1}{d}) < \alpha_1 < 2$ (the lower bound is empty unless $d = 3$). Then for any finite $\rho > 0$, there are constants $r > d$, $\lambda_0 > 0$, and $0 < c_1 \leq 1$, such that the following holds: For $\lambda_0 \leq \lambda < \infty$, there exist solutions of (1.1) of the form (3.1), with*

$$\sup_{t\geq 0}\left\{e^{\lambda t}\|\eta(t)\|_{L^2_x} + e^{c_1\lambda t}\|\nabla\eta(t)\|_{L^2_x\cap L^r_x}\right\} \leq \rho. \tag{3.30}$$

If moreover $d = 1$ and $\alpha_1 \geq 1$, then the above assertion holds with $r = \infty$.

Remark. We need $d \leq 3$ so that $\|e^{it\Delta}\|_{L^{r'}\to L^r}$ is locally integrable in t for some $r > d$. We need $\alpha_1 < 2$ so that $\mathcal{C}_B$ in (3.14) is nonempty, and hence B_p can be controlled for $p \in \mathcal{C}_B$.

Proof. For $r > d$, $\lambda > 0$, and $0 < c_1 \leq 1$, let $X = X_{r,\lambda,c_1}$ be the Banach space of all $\eta : [0,\infty)\times\mathbb{R}^d \to \mathbb{C}$ with norm $\|\eta\|_X$ defined by the left-hand side of (3.30). By the Gagliardo-Nirenberg's inequality, for any $p \in [2,\infty]$,

$$\|\eta(t)\|_{L^p_x} \leq G_{d,p,r}\|\eta(t)\|_{L^2_x}^{1-\theta}\|\nabla\eta(t)\|_{L^r_x}^{\theta} \leq G_{d,p,r}\|\eta\|_X e^{-(1-\theta+c_1\theta)\lambda t}, \tag{3.31}$$

where $G_{d,p,r}$ is a constant and $\theta = (\frac{1}{2}-\frac{1}{p})(\frac{1}{2}+\frac{1}{d}-\frac{1}{r})^{-1}$.

We will show that Φ can be a contraction mapping on the closed unit ball of X (the case of $\rho = 1$). Balls with other radius can be similarly treated. Moreover, we'll only give the estimates for Φ to be a self-mapping. As in the proof of Theorem 3.7, the derivations of the estimates for contractivity are no harder (and without the H parts).

Given $\eta \in X$ with $\|\eta\|_X \le 1$. We will first estimate $\|\Phi\eta(t)\|_{L^2_x}$, and then $\|\nabla\Phi\eta(t)\|_{L^r_x}$. Finally, $\|\nabla\Phi\eta(t)\|_{L^2_x}$ is basically a special case of $\|\nabla\Phi\eta(t)\|_{L^r_x}$.

Part 1. Estimate of $\|\Phi\eta(t)\|_{L^2_x}$. For G, we have

$$\begin{aligned}\|G(\tau)\|_{L^2_x} &\lesssim \sum_{i=1,2}(\||\eta||T|^{\alpha_i}(\tau)\|_{L^2_x} + \||\eta|^{\alpha_i+1}(\tau)\|_{L^2_x})\\ &\le \sum_{i=1,2}(\|\eta(\tau)\|_{L^2_x}\|T\|^{\alpha_i}_{L^\infty_x} + \|\eta(\tau)\|_{L^2_x}\|\eta(\tau)\|^{\alpha_i}_{L^\infty_x})\\ &\lesssim \sum_{i=1,2}(A^{\alpha_i}_\infty + G^{\alpha_i}_{d,\infty,r})e^{-\lambda\tau},\end{aligned} \tag{3.32}$$

by (3.31). Then consider H. Since $d \le 3$, we have $2 > \frac{d\alpha_1}{2(\alpha_1+1)}$ (for all $\alpha_1 > 0$). Fix any $2 > s_1 > \frac{d\alpha_1}{2(\alpha_1+1)}$, we get from Lemma 3.4 (H0)

$$\|H(\tau)\|_{L^2_x} \lesssim (\sum_{i=1,2} A^{\alpha_i+1}_{(\alpha_i+1)s_1})^{s_1/2}(\sum_{i=1,2} A^{\alpha_i+1}_\infty)^{1-s_1/2}e^{-a(1-s_1/2)v_*\tau}, \tag{3.33}$$

with $(\alpha_1+1)s_1 \in \mathcal{C}_A$. Suppose

$$a(1-s_1/2)v_* \ge \lambda.$$

Then from (3.32) and (3.33), the dispersive inequality gives

$$\|\Phi\eta(t)\|_{L^2_x} \lesssim \int_t^\infty E_1e^{-\lambda\tau}\,d\tau = E_1\lambda^{-1}e^{-\lambda t}, \tag{3.34}$$

where

$$E_1 = \sum_{i=1,2}(A^{\alpha_i}_\infty + G^{\alpha_i}_{d,\infty,r}) + (\sum_{i=1,2} A^{\alpha_i+1}_{(\alpha_i+1)s_1})^{s_1/2}(\sum_{i=1,2} A^{\alpha_i+1}_\infty)^{1-s_1/2}.$$

Part 2. Estimate of $\|\nabla\Phi\eta(t)\|_{L^r_x}$. This part is more delicate. The dispersive inequality gives

$$\|\nabla\Phi\eta(t)\|_{L^r_x} \lesssim \int_t^\infty |t-\tau|^{-d(\frac12-\frac1r)}(\|\nabla G(\tau)\|_{L^{r'}_x} + \|\nabla H(\tau)\|_{L^{r'}_x})\,d\tau.$$

To derive a suitable estimate from it, in the following we will get several conditions on the lower bounds of $1/r$. The one thing to check is that they are all strictly less than $1/d$, so that there is really one $r > d$ satisfying all the conditions. Moreover, if $d = 1$, r can be ∞.

Step 1. For $|t-\tau|^{-d(\frac12-\frac1r)}$ to be integrable at the singularity $\tau = t$, we need $d(\frac12-\frac1r) < 1$, i.e.

$$\frac12 - \frac1d < \frac1r. \tag{Condition 1}$$

Since we want $r > d$, we need the lower bound $\frac12 - \frac1d$ to be less than $\frac1d$, which holds if and only if $d \le 3$. If $d = 1$, the lower bound is negative and we can choose $r = \infty$.

Step 2. Estimate of G. By (2.4),

$$|\nabla G| = |\nabla[f(T+\eta) - f(T)]|$$
$$\lesssim \sum_{i=1,2} \left\{ |\eta|^{\min(\alpha_i,1)}(|T|+|\eta|)^{\max(\alpha_i-1,0)}|\nabla T| + (|T|+|\eta|)^{\alpha_i}|\nabla \eta| \right\}.$$

If $\alpha_i > 1$, we have to estimate the $L_x^{r'}$ norm of (1) $|\eta||T|^{\alpha_i-1}|\nabla T|$, (2) $|\eta|^{\alpha_i}|\nabla T|$, (3) $|T|^{\alpha_i}|\nabla \eta|$, and (4) $|\eta|^{\alpha_i}|\nabla \eta|$. And if $\alpha_i \le 1$, we only have to estimate (2), (3) and (4). We discuss them in the following. We remark that the p, q in different sub-steps are unrelated.

Step 2-1. Estimate of (1). (Only for $\alpha_i > 1$) Suppose

$$\frac{1}{r'} < \frac{1}{2} + \frac{2(\alpha_i - 1)}{d\alpha_1} + \frac{2}{d}\left(\frac{1}{\alpha_1} - \frac{1}{2}\right). \tag{3.35}$$

Then, since $\frac{1}{2} \le \frac{1}{r'}$, we have $\frac{1}{r'} = \frac{1}{2} + \frac{1}{q} + \frac{1}{p}$ for some $p, q \in (0, \infty]$ satisfying $\frac{1}{q} < \frac{2(\alpha_i-1)}{d\alpha_1}$ and $\frac{1}{p} < \frac{2}{d}(\frac{1}{\alpha_1} - \frac{1}{2})$. Thus

$$\begin{aligned} \| |\eta||T|^{\alpha_i-1}|\nabla T|(\tau)\|_{L_x^{r'}} &\le \|\eta(\tau)\|_{L_x^2} \||T|^{\alpha_i-1}\|_{L_x^q} \|\nabla T\|_{L_x^p} \\ &\lesssim A^{\alpha_i-1}_{(\alpha_i-1)q} B_p e^{-\lambda\tau} \\ &\le A^{\alpha_i-1}_{(\alpha_i-1)q} B_p e^{-c_1\lambda\tau}, \end{aligned} \tag{3.36}$$

with $(\alpha_i - 1)q \in \mathcal{C}_A$ and $p \in \mathcal{C}_B$. Notice that (3.35) is equivalent to

$$\frac{1}{2} - \frac{2}{d}\left(\frac{\alpha_i}{\alpha_1} - \frac{1}{2}\right) < \frac{1}{r}. \tag{Condition 2}$$

It's easy to check that, since $d \le 3$ and $\alpha_i \ge \alpha_1$, the lower bound is strictly less than $1/d$, and is negative if $d = 1$.

Step 2-2. Estimate of (2). Let $q = \max(r', 2/\alpha_1)$, and p be such that $\frac{1}{r'} = \frac{1}{q} + \frac{1}{p}$. By (3.31)

$$\begin{aligned} \| |\eta|^{\alpha_i}|\nabla T|(\tau)\|_{L_x^{r'}} &\le \|\eta(\tau)\|^{\alpha_i}_{L_x^{\alpha_i q}} \|\nabla T\|_{L_x^p} \\ &\lesssim B_p G^{\alpha_i}_{d,\alpha_i q, r} e^{-\alpha_i(1-\theta+c_1\theta)\lambda\tau}, \end{aligned} \tag{3.37}$$

where $\theta = (\frac{1}{2} - \frac{1}{\alpha_i q})(\frac{1}{2} + \frac{1}{d} - \frac{1}{r})^{-1}$. For (3.37) to be an effective estimate, we need (i) $p \in \mathcal{C}_B$, and (ii) $\alpha_i(1 - \theta + c_1\theta) \ge c_1$.

Since $\frac{1}{p} = \frac{1}{r'} - \frac{1}{q}$, (i) holds iff

$$\frac{1}{r'} - \frac{1}{q} < \frac{2}{d}\left(\frac{1}{\alpha_1} - \frac{1}{2}\right). \tag{3.38}$$

There are two cases according to the value of q. If $q = r'$ (i.e. $r' \ge 2/\alpha_1$), then (3.38) is automatically true since $\alpha_1 < 2$, and no restriction on r is needed. On the other hand, if $q = 2/\alpha_1$ (i.e. $r' < 2/\alpha_1$), then (3.38) gives

$$1 - \left(\frac{\alpha_1}{2} + \frac{2}{d\alpha_1}\right) + \frac{1}{d} < \frac{1}{r}. \tag{Condition 3}$$

Since

$$\frac{\alpha_1}{2} + \frac{2}{d\alpha_1} \geq \frac{2}{\sqrt{d}}, \tag{3.39}$$

the lower bound is less than $1/d$ for $d \leq 3$. Moreover, if $d = 1$, strict inequality holds in (3.39), and the lower bound is negative.

Now consider (ii). Since $r > d$, there exists some $c_M = c_M(d, \alpha_1, \alpha_2, r) > 0$ such that (ii) holds as long as $0 < c_1 \leq c_M$. For example, we can use the (rather rough) estimate

$$\alpha_i(1 - \theta + c_1\theta) \geq \alpha_1(1 - \theta).$$

Hence (ii) holds if

$$0 < c_1 \leq \alpha_1 \left(1 - \frac{1}{2}\Big(\frac{1}{2} + \frac{1}{d} - \frac{1}{r}\Big)^{-1}\right). \tag{3.40}$$

Step 2-3. Estimate of (3). We have

$$\||T|^{\alpha_i}|\nabla\eta|(\tau)\|_{L_x^{r'}} \leq \||T|^{\alpha_i}\|_{L_x^q}\|\nabla\eta(\tau)\|_{L_x^2} \lesssim A_{\alpha_i q}^{\alpha_i} e^{-c_1\lambda\tau}, \tag{3.41}$$

where $\frac{1}{r'} = \frac{1}{q} + \frac{1}{2}$. We need $\alpha_i q \in \mathcal{C}_A$, i.e.

$$\frac{1}{2} - \frac{2\alpha_i}{d\alpha_1} < \frac{1}{r}. \tag{Condition 4}$$

The lower bound is less than $1/d$, and is negative if $d = 1$.

Step 2-4. Estimate of (4). We have

$$\||\eta|^{\alpha_i}|\nabla\eta|(\tau)\|_{L_x^{r'}} \leq \|\eta(\tau)\|_{L_x^{\alpha_i q}}^{\alpha_i}\|\nabla\eta(\tau)\|_{L_x^2},$$

where $\frac{1}{r'} = \frac{1}{q} + \frac{1}{2}$. If $\alpha_i q \geq 2$, that is

$$\frac{1}{2} - \frac{\alpha_i}{2} \leq \frac{1}{r}, \tag{Condition 5}$$

then we get from (3.31)

$$\||\eta|^{\alpha_i}|\nabla\eta|(\tau)\|_{L_x^{r'}} \leq G_{d,\alpha_i q,r}^{\alpha_i} e^{-c_1\lambda\tau}. \tag{3.42}$$

The lower bound in (Condition 5) is less than $1/d$ since $2(\frac{1}{2} - \frac{1}{d}) < \alpha_1$ (this is where we need this requirement). Moreover, r can be ∞ if $\alpha_1 \geq 1$.

Step 3. Estimate of H. Suppose

$$\frac{1}{r'} < \frac{2}{d} + \frac{2}{d}(\frac{1}{\alpha_1} - \frac{1}{2}), \tag{3.43}$$

equivalently

$$1 - \frac{2}{d}\left(\frac{1}{\alpha_1} + \frac{1}{2}\right) < \frac{1}{r}. \tag{Condition 6}$$

One can check that the lower bound is less than $1/d$ by $d \leq 3$ and $\alpha_1 < 2$, and is negative if $d = 1$. From (3.43), by fixing a small enough $\varepsilon > 0$, we have

$$\frac{1}{q} := \frac{2}{d} - \varepsilon \geq 0, \quad \frac{1}{p} := \frac{2}{d}(\frac{1}{\alpha_1} - \frac{1}{2}) - \varepsilon \geq 0,$$

and
$$\frac{1}{s_2} := \frac{1}{p} + \frac{1}{q} = \frac{2}{d} + \frac{2}{d}(\frac{1}{\alpha_1} - \frac{1}{2}) - 2\varepsilon > \frac{1}{r'}.$$
From Lemma 3.4 (H1), we get
$$\|\nabla H(\tau)\|_{L_x^{r'}} \lesssim (\sum_{i=1,2} A_{\alpha_i q}^{\alpha_i} B_p)^{s_2/r'} (\sum_{i=1,2} A_\infty^{\alpha_i} B_\infty)^{1-s_2/r'} e^{-a\min(\alpha_1,1)(1-s_2/r')v_*\tau}. \tag{3.44}$$

Since $\frac{1}{q} < \frac{2}{d}$ and $\frac{1}{p} < \frac{2}{d}(\frac{1}{\alpha_1} - \frac{1}{2})$, we have $\alpha_1 q \in \mathcal{C}_A$ and $p \in \mathcal{C}_B$.

From the above discussions, we get the following conclusion: Suppose $r > d$ is sufficiently close to d, c_1 satisfies (3.40), and v_* satisfies
$$a\min(\alpha_1, 1)(1 - s_2/r')v_* \geq c_1\lambda. \tag{3.45}$$

Then we have
$$\begin{aligned}\|\nabla\Phi\eta(t)\|_{L_x^r} &\lesssim \int_t^\infty |t-\tau|^{-d(\frac{1}{2}-\frac{1}{r})} E_2 e^{-c_1\lambda\tau}\, d\tau \\ &= E_2\Gamma(1 - d(\frac{1}{2} - \frac{1}{r}))(c_1\lambda)^{d(\frac{1}{2}-\frac{1}{r})-1} e^{-c_1\lambda t},\end{aligned} \tag{3.46}$$

where E_2 is obtained by collecting the coefficients in (3.36), (3.37), (3.41), (3.42), and (3.44).

Part 3. Estimate of $\|\nabla\Phi\eta(t)\|_{L_x^2}$. We have
$$\|\nabla\Phi\eta(t)\|_{L_x^2} \lesssim \int_t^\infty (\|\nabla G(\tau)\|_{L_x^2} + \|\nabla H(\tau)\|_{L_x^2})\, d\tau.$$

We can imitate Part 2 to obtain all the needed estimates. We summarize them below.

1. There is no need of Step 1.
2. For the four sub-steps in Step 2, simply replace "r" by "2" (except those of $G_{d,p,r}$ and θ in using (3.31)), we have the following results:
   - (2-1) $\||\eta||T|^{\alpha_i-1}|\nabla T|(\tau)\|_{L_x^2} \lesssim A_\infty^{\alpha_i-1} B_\infty e^{-c_1\lambda\tau}$.
   - (2-2) $\||\eta|^{\alpha_i}|\nabla T|(\tau)\|_{L_x^2} \lesssim B_p G_{d,\alpha_i q,r}^{\alpha_i} e^{-\alpha_i(1-\theta+c_1\theta)\lambda\tau}$, where $q = \max(2, 2/\alpha_1)$, p is such that $\frac{1}{2} = \frac{1}{q} + \frac{1}{p}$, and $\theta = (\frac{1}{2} - \frac{1}{\alpha_i q})(\frac{1}{2} + \frac{1}{d} - \frac{1}{r})^{-1}$. It's easy to check that $p \in \mathcal{C}_B$, and $\alpha_i(1 - \theta + c_1\theta) \geq c_1$ as long as (3.40) holds.
   - (2-3) $\||T|^{\alpha_i}|\nabla\eta|(\tau)\|_{L_x^2} \lesssim A_\infty^{\alpha_i} e^{-c_1\lambda\tau}$.
   - (2-4) $\||\eta|^{\alpha_i}|\nabla\eta|(\tau)\|_{L_x^2} \leq G_{d,\infty,r}^{\alpha_i} e^{-c_1\lambda\tau}$.
3. The conclusion of Step 3 is valid with r replaced by 2. Precisely, we have
$$\|\nabla H(t)\|_{L_x^2} \lesssim (\sum_{i=1,2} A_{\alpha_i q}^{\alpha_i} B_p)^{s_2/2} (\sum_{i=1,2} A_\infty^{\alpha_i} B_\infty)^{s_2/2} e^{-a\min(\alpha_1,1)(1-s_2/2)v_* t}, \tag{3.47}$$
   where s_2, p, q can be the same as given there.

Thus, if (3.40) and (3.45) hold, we have

$$\|\nabla\Phi\eta(t)\|_{L_x^2} \lesssim \int_t^\infty E_3 e^{-c_1\lambda\tau}\, d\tau = E_3(c_1\lambda)^{-1}e^{-c_1\lambda t}, \tag{3.48}$$

where E_3 is obtained by collecting the coefficients in (2-1) – (2-4) and (3.47).

The conclusions of the three Parts (namely (3.34), (3.46) and (3.48)) provide the needed estimates for Φ to be a self-mapping. Similarly we can derive the estimates for Φ to be contractive, and the theorem is true by Lemma 3.6. □

Remark 3.10. Our assertion will be weaker without considering the $\|\nabla\eta(t)\|_{L_x^2}$ control. Precisely, without it, due to the necessary modification of *Step 2-4* in Part 2, (Condition 5) becomes $\frac{1}{2} - \frac{\alpha_i}{4} \leq \frac{1}{r}$. Thus we need $4(\frac{1}{2} - \frac{1}{d}) < \alpha_1$ (for $\frac{1}{r} < \frac{1}{d}$ to be possible). Also, since $\alpha_1 < 2$, $r = \infty$ is not allowed.

By Theorem 3.7, Theorem 3.9, and the Gagliardo-Nirenberg's inequality (3.31), we have proved the following

Corollary 3.11. *Let $d \leq 3$, and f satisfy Assumptions (F) and (T)$_d$. Assume moreover either of the following conditions:*

(i) $0 < \alpha_1 \leq \alpha_2 < \infty$ *if* $d = 1$.

(ii) $2(\frac{1}{2} - \frac{1}{d}) < \alpha_1 < 2$ *if* $d = 2, 3$.

Then for any finite $\rho > 0$, there exists a constant $\lambda_0 > 0$ such that the following holds: For $\lambda_0 \leq \lambda < \infty$, there exist solutions of (1.1) *of the form* (3.1), *with*

$$\sup_{t\geq 0} e^{\lambda t}\|\eta(t)\|_{L_x^2\cap L_x^\infty} \leq \rho.$$

4 Mixed dimensional trains

In this section we consider mixed trains. It would be good for the reader to recall the discussion in Section 1.3.

First we point out a new problem not mentioned in Section 1.3: We can't use only the dispersive inequality (3.15) to construct mixed trains like we did in the previous section. To explain the problem, we take the 1D-2D train $T_1 + \eta_1 + T_2 + \eta$ for example. Corresponding to this train, we have

$$G = f(T_1 + \eta_1 + T_2 + \eta) - f(T_1 + \eta_1 + T_2).$$

Suppose we try to find η in a Banach space X whose norm assumes the exponential decay of $\|\eta(t)\|_{L_x^p}$ (with possibly several p). Then we have to estimate $\|\Phi\eta(t)\|_{L_x^p}$. To use the dispersive inequality, we can only consider $p \geq 2$. Then we have to estimate $\|G(\tau)\|_{L_x^{p'}}$, from which we will encounter (a) $\||\eta|(|T_1| + |\eta_1|)^{\alpha_i}(\tau)\|_{L_x^{p'}}$ and (b) $\||\eta|^{\alpha_i+1}(\tau)\|_{L_x^{p'}}$ (and also $\||\eta||T_2|^{\alpha_i}(\tau)\|_{L_x^{p'}}$, which is not relevant to the problem). For (a), since the 1D objects only have L^∞ bounds in x_2, not $L_{x_2}^q$ for $q < \infty$, we can only estimate as follows:

$$\||\eta|(|T_1| + |\eta_1|)^{\alpha_i}(\tau)\|_{L_x^{p'}} \leq \|\eta(\tau)\|_{L_x^{p'}}\|(|T_1| + |\eta_1|)(\tau)\|_{L_x^\infty}^{\alpha_i}.$$

Thus we have to also assume the exponential decay of $\|\eta(t)\|_{L_x^{p'}}$ for the norm of X, and hence have to estimate $\|\Phi\eta(t)\|_{L_x^{p'}}$. Again, this can be done only if $p' \geq 2$, and hence we must have $p = p' = 2$. Nevertheless, (b) then requires us to estimate $\|\eta(\tau)\|_{L_x^{2(\alpha_i+1)}}$, and the construction fails. We remark that adding some $\|\nabla\eta(t)\|_{L_x^p}$ controls in the definition of X also results in similar problems.

Due to the above observation, we shall use the Strichartz estimate to accomplish our task. In the following section, we recall the basic definitions and facts about the Strichartz space, and then give some more specialized inequalities to be used.

4.1 Strichartz space

Let $\mathcal{A} = \mathcal{A}^{(d)}$ be the set of all pairs (q, r) satisfying $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$, with $2 \leq r \leq r_{\max}$ or equivalently $q_{\min} \leq q \leq \infty$, where

$$r_{\max} = r_{\max}^{(d)} = \begin{cases} \infty & \text{if } d = 1 \\ 4 & \text{if } d = 2 \\ \frac{2d}{d-2} & \text{if } d \geq 3, \end{cases} \quad \text{and} \quad q_{\min} = q_{\min}^{(d)} = \begin{cases} 4 & \text{if } d = 1 \\ 4 & \text{if } d = 2 \\ 2 & \text{if } d \geq 3. \end{cases} \tag{4.1}$$

Thus $\mathcal{A}$ is the set of all (Schrödinger) admissible pairs if $d \neq 2$. For $d = 2$, we take $r_{\max} < \infty$ to avoid the forbidden endpoint, and $r_{\max}$ can actually be any finite number no less than 4 for our approach. We set it to be 4 for preciseness.

For $\tau \geq 0$, we abbreviate $L^q([\tau, \infty), L^r(\mathbb{R}^d))$ as $L_t^q L_x^r(\tau)$, or even $L_t^q L_x^r$ when the time interval is clear. We'll abuse notation and write $L_t^q L_x^r(t)$, where the two "t" should not cause confusion. Define the Strichartz space

$$S(t) := L_t^\infty L_x^2(t) \cap L_t^{q_{\min}} L_x^{r_{\max}}(t),$$

with norm

$$\|\cdot\|_{S(t)} := \max(\|\cdot\|_{L_t^\infty L_x^2(t)}, \|\cdot\|_{L_t^{q_{\min}} L_x^{r_{\max}}(t)}).$$

By interpolation,

$$S(t) = \bigcap_{(q,r)\in\mathcal{A}} L_t^q L_x^r(t), \quad \text{and} \quad \|\cdot\|_{S(t)} = \sup_{(q,r)\in\mathcal{A}} \|\cdot\|_{L_t^q L_x^r(t)}.$$

Denote the dual space of $S(t)$ by $N(t)$. For $(q, r) \in \mathcal{A}$, a function $\xi \in L_t^{q'} L_x^{r'}(t)$ is regarded as an element in $N(t)$ by letting

$$\langle \xi, \eta \rangle_{N(t),S(t)} := \int_t^\infty \int_{\mathbb{R}^d} \xi(s, x)\eta(s, x)\, dxds \quad (\text{for } \eta \in S(t)).$$

In this way, we have $|\langle \xi, \eta \rangle_{N(t),S(t)}| \leq \|\xi\|_{L_t^{q'} L_x^{r'}(t)} \|\eta\|_{S(t)}$, and hence

$$\|\xi\|_{N(t)} \leq \|\xi\|_{L_t^{q'} L_x^{r'}(t)}.$$

For $\lambda > 0$ and $t_0 \geq 0$, we define S_{λ,t_0} to be the class of all $\eta \in S(t_0)$ such that

$$\|\eta\|_{S_{\lambda,t_0}} := \sup_{t \geq t_0} e^{\lambda t} \|\eta\|_{S(t)} < \infty.$$

By definition, $\|\eta\|_{S(t)} \le \|\eta\|_{S_{\lambda,t_0}} e^{-\lambda t}$ for $t \ge t_0$. In particular, since $\|\eta\|_{L_t^\infty L_x^2(t)} \le \|\eta\|_{S(t)}$, we have

$$\|\eta(t)\|_{L_x^2} \le \|\eta\|_{S_{\lambda,t_0}} e^{-\lambda t} \quad \text{for (almost all)} \quad t \ge t_0. \tag{4.2}$$

In the rest of this section we prove some useful inequalities, particularly Lemma 4.4. First, we give a fact arising from a proof step of [5, Proposition 2.4]. It might be of independent interest.

Proposition 4.1. *Given $0 < p \le q \le \infty$ and $\lambda > 0$. If $u : [0,\infty) \to [0,\infty]$ satisfies $\|u\|_{L^q([t,\infty))} \le e^{-\lambda t}$ for all $t \ge 0$, then*

$$\|u\|_{L^p([t,\infty))} \le \widetilde{C} \lambda^{\frac{1}{q}-\frac{1}{p}} e^{-\lambda t} \quad \forall\, t \ge 0, \tag{4.3}$$

where we can choose $\widetilde{C} = \widetilde{C}(p)$ in such a way that $\widetilde{C} \le (1-e^{-1})^{-1}$ for $p \ge 1$.

Proof. We consider three cases separately.

1. If $p = q$, (4.3) is trivially true with $\widetilde{C} = 1$.

2. If $p < q = \infty$, we have $u(t) \le e^{-\lambda t}$ for a.a. $t \in [0,\infty)$, and hence

$$\|u\|^p_{L^p([t,\infty))} \le \int_t^\infty e^{-p\lambda\tau}\, d\tau = (p\lambda)^{-1} e^{-p\lambda t}.$$

   So (4.3) is true with $\widetilde{C} = p^{-1/p}$.

3. Suppose $p < q < \infty$. For fixed $t \ge 0$, let $\{t_k\}_{k=0}^\infty$ be a sequence satisfying $t_0 = t$ and $t_k \nearrow \infty$. Then

$$\begin{aligned}
\|u\|^p_{L^p([t,\infty))} &= \sum_{k=0}^\infty \int_{t_k}^{t_{k+1}} u(\tau)^p\, d\tau \\
&\le \sum_{k=0}^\infty \Big(\int_{t_k}^{t_{k+1}} u(\tau)^q\, d\tau \Big)^{p/q} (t_{k+1}-t_k)^{1-p/q} \\
&\le \sum_{k=0}^\infty \|u\|^p_{L^q([t_k,\infty))} (t_{k+1}-t_k)^{1-p/q} \\
&\le \sum_{k=0}^\infty e^{-p\lambda t_k} (t_{k+1}-t_k)^{1-p/q}.
\end{aligned}$$

   Letting $t_k = t + \frac{k}{\lambda}$, we get

$$\|u\|^p_{L^p([t,\infty))} \le (1-e^{-p})^{-1} \lambda^{p/q-1} e^{-p\lambda t}.$$

   Thus (4.3) is true with $\widetilde{C} = (1-e^{-p})^{-1/p}$.

Comparing the three cases, we see $\widetilde{C} \le (1-e^{-1})^{-1}$ for $p \ge 1$. □

Definition 4.2. If $(q,r) \in \mathcal{A}$ and $0 < p \le q$, we call (p,r) *sub-admissible.* Thus (p,r) is sub-admissible if and only if $2 \le r \le r_{\max}$, $p > 0$, and $\frac{2}{p} + \frac{d}{r} \ge \frac{d}{2}$.

Corollary 4.3. *Let $\eta \in S_{\lambda,t_0}$. If $(q,r) \in \mathcal{A}$ and (p,r) is sub-admissible, then*

$$\|\eta\|_{L_t^p L_x^r(t)} \lesssim \lambda^{\frac{1}{q}-\frac{1}{p}} \|\eta\|_{S_{\lambda,t_0}} e^{-\lambda t} = \lambda^{\frac{1}{2}(\frac{d}{2}-\frac{d}{r}-\frac{2}{p})} \|\eta\|_{S_{\lambda,t_0}} e^{-\lambda t} \quad (t \ge t_0).$$

Proof. The case of $\eta = 0$ is trivial. Assume $\eta \neq 0$. Define $u : [0,\infty) \to [0,\infty]$ by

$$u(t) = \frac{e^{\lambda t_0}}{\|\eta\|_{S_{\lambda,t_0}}} \|\eta(t+t_0)\|_{L_x^r},$$

then $\|u\|_{L^q([t,\infty))} \le e^{-\lambda t}$ for $t \ge 0$. By Proposition 4.1, $\|u\|_{L^p([t,\infty))}$ satisfies (4.3), which gives what we want to show. ☐

Definition 4.2 and Corollary 4.3 are only used in the next result.

Lemma 4.4. *We have the following estimates.*

(N0) *Suppose $0 \le m \le 4/d$. For $u, v \in S_{\lambda,t_0}$,*

$$\||u||v|^m\|_{N(t)} \lesssim \lambda^{-1+dm/4} \|u\|_{S_{\lambda,t_0}} \|v\|_{S_{\lambda,t_0}}^m e^{-(m+1)\lambda t} \quad \forall t \ge t_0.$$

(N1) *Suppose $0 \le m < \alpha_{\max}$. For $u, v \in S_{\lambda,t_0}$ such that $|\nabla v| \in S_{\lambda,t_0}$,*

$$\||u||v|^m\|_{N(t)} \lesssim_{d,m} \lambda^{-\mu} \|u\|_{S_{\lambda,t_0}} \|v\|_{S_{\lambda,t_0}}^{m(1-b)} \|\nabla v\|_{S_{\lambda,t_0}}^{mb} e^{-(m+1)\lambda t} \quad \forall t \ge t_0,$$

for some $b = b(d,m) \in [0,1]$ and $\mu = \mu(d,m) > 0$ (given explicitly in the proof).

Proof. Consider (N0). For $(q,r) \in \mathcal{A}$,

$$\||u||v|^m\|_{N(t)} \le \||u||v|^m\|_{L_t^{q'} L_x^{r'}(t)} \le \|u\|_{L_t^{(m+1)q'} L_x^{(m+1)r'}(t)} \|v\|_{L_t^{(m+1)q'} L_x^{(m+1)r'}(t)}^m. \tag{4.4}$$

We want to show that there is $(q,r) \in \mathcal{A}$ such that $((m+1)q', (m+1)r')$ is sub-admissible. That is, there are q, r such that the following conditions hold ((i),(ii) $\Leftrightarrow$ $(q,r) \in \mathcal{A}$; (iii),(iv) $\Leftrightarrow$ $((m+1)q', (m+1)r')$ is sub-admissible):

(i) $\frac{2}{q'} + \frac{d}{r'} = 2 + \frac{d}{2}$.

(ii) $r'_{\max} \le r' \le 2$.

(iii) $\frac{2}{(m+1)q'} + \frac{d}{(m+1)r'} \ge \frac{d}{2}$.

(iv) $2 \le (m+1)r' \le r_{\max}$.

It's enough to prove the existence of r' satisfying (ii) and (iv), since (ii) implies the existence of q such that (i) holds, and then (iii) also holds by $m \le 4/d$. Now (ii) and (iv) are satisfied by some r' if and only if

$$\frac{2}{m+1} \le 2 \quad \text{and} \quad r'_{\max} \le \frac{r_{\max}}{m+1},$$

that is

$$1 \le m+1 \le \frac{r_{\max}}{r'_{\max}} = \begin{cases} \infty & \text{if } d=1 \\ 3 & \text{if } d=2 \\ \frac{d+2}{d-2} & \text{if } d \ge 3. \end{cases} \tag{4.5}$$

Since $0 \le m \le 4/d$, (4.5) is satisfied. (This is where we need $r_{\max} \ge 4$ for $d = 2$.) Now let $(q, r) \in \mathcal{A}$ be such that $((m+1)q', (m+1)r')$ is sub-admissible, then (4.4) and Corollary 4.3 imply

$$\| |u||v|^m \|_{N(t)} \le \widetilde{C}^{m+1} \lambda^{-1+dm/4} \|u\|_{S_{\lambda,t_0}} \|v\|^m_{S_{\lambda,t_0}} e^{-(m+1)\lambda t}.$$

This proves (N0).

Now we consider (N1). The case of $m = 0$ is justified in (N0), so assume $m > 0$. Then, for $(q, r) \in \mathcal{A}$ and any $0 \le \theta, \phi \le 1$,

$$\| |u||v|^m \|_{N(t)} \le \| |u||v|^m \|_{L_t^{q'} L_x^{r'}(t)} \le \|u\|_{L_t^{q'/(1-\theta)} L_x^{r'/(1-\phi)}(t)} \|v\|^m_{L_t^{mq'/\theta} L_x^{mr'/\phi}(t)}. \tag{4.6}$$

By the Gagliardo-Nirenberg's inequality, if p, b are two numbers satisfying $p \ge 1$,

$$0 \le b \le 1, \ b \ne 1 \text{ if } p = d > 1, \tag{b1}$$

and

$$\frac{\phi}{mr'} = \frac{1}{p} - \frac{b}{d}, \tag{4.7}$$

then we have

$$\begin{aligned} \|v\|_{L_t^{mq'/\theta} L_x^{mr'/\phi}} &= \left\| \|v\|_{L_x^{mr'/\phi}} \right\|_{L_t^{mq'/\theta}} \\ &\lesssim_{d,p,b} \left\| \|\nabla v\|^b_{L_x^p} \|v\|^{1-b}_{L_x^p} \right\|_{L_t^{mq'/\theta}} \\ &\le \left\| \|\nabla v\|^b_{L_x^p} \right\|_{L_t^{mq'/(\theta b)}} \left\| \|v\|^{1-b}_{L_x^p} \right\|_{L_t^{mq'/(\theta(1-b))}} \\ &= \|\nabla v\|^b_{L_t^{mq'/\theta} L_x^p} \|v\|^{1-b}_{L_t^{mq'/\theta} L_x^p}. \end{aligned} \tag{4.8}$$

Consider $p \ge 2$. Let $\theta = m/(m+1)$ and $\phi = 1 - r'/p$ (which lies in $[0, 1]$ since $p \ge 2$), then (4.6) and (4.8) give

$$\| |u||v|^m \|_{N(t)} \lesssim_{d,p,b} \|u\|_{L^{(m+1)q'} L^p(t)} \|v\|^{m(1-b)}_{L^{(m+1)q'} L^p(t)} \|\nabla v\|^{mb}_{L^{(m+1)q'} L^p(t)}, \tag{4.9}$$

where

$$\frac{1}{r'} = \frac{m+1}{p} - \frac{mb}{d} \tag{4.10}$$

from (4.7). In summary, for the validity of (4.9), we need (i) $(q, r) \in \mathcal{A}$, (ii) $p \geq 2$, (iii) (b1) holds, and (iv) (4.10) holds. For the existence of such q, r, p, b, it suffices to show that there exist p, b satisfying (ii), (iii), and

$$\frac{1}{2} \leq \frac{m+1}{p} - \frac{mb}{d} \leq \frac{1}{r'_{\max}}. \tag{4.11}$$

Since then, by defining r' by (4.10) (hence (iv) holds), there is q such that (i) holds. Notice that (4.11) is equivalent to

$$\frac{d}{m}\Big(\frac{m+1}{p} - \frac{1}{r'_{\max}}\Big) \leq b \leq \frac{d}{m}\Big(\frac{m+1}{p} - \frac{1}{2}\Big). \tag{b2}$$

Hence we need $p \geq 2$ and b satisfying (b1) and (b2). Moreover, we want to choose p, b so that $((m+1)q', p)$ is "strictly" sub-admissible, i.e. $2 \leq p \leq r_{\max}$ and

$$\mu := \frac{m+1}{2}\Big(\frac{d}{p} + \frac{2}{(m+1)q'} - \frac{d}{2}\Big) = 1 - \frac{m}{2}\big(\frac{d}{2} - b\big) > 0, \tag{4.12}$$

where for the equality we use $(q, r) \in \mathcal{A}$ and (4.10). Once we have such p, b, then by Corollary 4.3, (4.9) gives

$$\||u||v|^m\|_{N(t)} \lesssim_{d,m} \widetilde{C}^{m+1}\lambda^{-\mu}\|u\|_{S_{\lambda,t_0}}\|v\|^{m(1-b)}_{S_{\lambda,t_0}}\|\nabla v\|^{mb}_{S_{\lambda,t_0}} e^{-(m+1)\lambda t},$$

which is exactly what we want to prove.

We give possible choices of p, b in the following. Notice that $\mu > 0$ is trivial if $d \leq 2$.

1. If $d = 1$, we can choose $p = 2$, and b any number satisfying $\max(\frac{m-1}{2m}, 0) \leq b \leq \frac{1}{2}$.
2. If $d = 2$, we can choose $p = 2$, and b any number satisfying $\max(\frac{2m-1}{2m}, 0) \leq b < 1$.
3. If $d \geq 3$, there exists b satisfying (b1) and (b2) if

$$\frac{2(m+1)d}{2(m+1)+d} \leq p \leq 2(m+1),$$

where the lower bound might be larger than 2. We consider two cases:

   (a) If $0 < m \leq \frac{2}{d-2}$, we can choose $p = 2$, and b any number satisfying $\max(\frac{d}{2} - \frac{1}{m}, 0) \leq b \leq 1$. One has $\mu \geq 1 - \frac{m}{2}(\frac{d}{2} - (\frac{d}{2} - \frac{1}{m})) = 1/2$.

   (b) If $\frac{2}{d-2} < m < \alpha_{\max} = \frac{4}{d-2}$, we can choose $p = \frac{2(m+1)d}{2(m+1)+d}$, and $b = 1$ (the only choice). It is easy to check that $2 \leq p \leq r_{\max}$ and $\mu > 0$. $\square$

4.2 Construction of eD-dD trains

Consider $1 \le e < d$. Let $x = (x', x'')$, where $x' = (x_1, \dots, x_e)$ and $x'' = (x_{e+1}, \dots, x_d)$. In this subsection we construct mixed dimensional soliton trains of the form

$$u = T_e + \eta_e + T_d + \eta, \tag{4.13}$$

where

$$T_e = \sum_{k \in \mathbb{N}} R_{e;k}(t, x'), \quad T_d = \sum_{j \in \mathbb{N}} R_{d;j}(t, x),$$

with $R_{e;k}$ and $R_{d;j}$ being eD and dD solitons as given by (1.5), with initial positions assumed to be the origin for simplicity. (The reservation of j for the indices of the dD solitons and k for those of the eD solitons will be convenient.) The eD error $\eta_e = \eta_e(t, x')$ is such that $T_e + \eta_e$ is itself an eD train (solution of (1.1)), whose existence will be provided by the previous section. And $\eta = \eta(t, x)$ is the remaining error to be found.

Denote the frequencies of $R_{d;j}$ and $R_{e;k}$ by ω_j and σ_k; and the velocities by

$$v_j = (v_{j,1}, v_{j,2}, \dots, v_{j,d}) \quad \text{and} \quad u_k = (u_{k,1}, \dots, u_{k,e}).$$

(Their corresponding bound states and phases will not be used explicitly, and hence there is no need to introduce notations for them.) $R_{e;k}$ is naturally regarded as a lower dimensional soliton in $\mathbb{R}^d_x$ by considering $R_{e;k}(t, x) \equiv R_{e;k}(t, x')$, with velocity $(u_{k,1}, \dots, u_{k,e}, 0, \dots, 0)$.

Besides the above, some more modifications of notation given in the previous section have to be made, and some anisotropic generalizations need to be introduced. We summarize them in the following.

1. We'll write $A_{d;p}$ for $A_p(\{\omega_j\})$ and $B_{d;p}$ for $B_p(\{\omega_j\}, \{v_j\})$, as defined in (3.4). Similarly, we write

$$A_{e;p} = A_{e;p}(\{\sigma_k\}) = \Big(\sum_k \sigma_k^{\min(1,p)(\frac{1}{\alpha_1} - \frac{e}{2p})}\Big)^{\max(1,p^{-1})}$$
$$B_{e;p} = B_{e;p}(\{\sigma_k\}, \{u_k\}) = \Big(\sum_k \langle u_k \rangle^{\min(1,p)} \sigma_k^{\min(1,p)(\frac{1}{\alpha_1} - \frac{e}{2p})}\Big)^{\max(1,p^{-1})}.$$

2. For $0 < p, q \le \infty$, we abbreviate the space $L^p(\mathbb{R}^e, L^q(\mathbb{R}^{d-e}))$ as $L^p_{x'} L^q_{x''}$. Recall that, for $u : \mathbb{R}^d \to \mathbb{C}$,

$$\|u\|_{L^p_{x'} L^q_{x''}} := \Big\| \|u(x', x'')\|_{L^q_{x''}} \Big\|_{L^p_{x'}} = \left(\int_{\mathbb{R}^e} \left(\int_{\mathbb{R}^{d-e}} |u(x', x'')|^q \, dx'' \right)^{p/q} dx' \right)^{1/p}.$$

In particular $L^p_x = L^p_{x'} L^p_{x''}$ with exactly the same norm. The following generalizations are straightforward, hence we only give them without proof. We have

$$\|R_{d;j}\|_{L^p_{x'} L^q_{x''}} \le D_{p,q} \omega_j^{\frac{1}{\alpha_1} - \frac{e}{2p} - \frac{d-e}{2q}}, \quad \text{and} \quad \|\nabla R_{d;j}\|_{L^p_{x'} L^q_{x''}} \le D_{p,q} \langle v_j \rangle \omega_j^{\frac{1}{\alpha_1} - \frac{e}{2p} - \frac{d-e}{2q}},$$

where $D_{p,q} = D\|e^{-a|y|}\|_{L^p_{y'}L^q_{y''}} \le D(\frac{2\sqrt{d}}{ap})^{e/p}(\frac{2\sqrt{d}}{aq})^{(d-e)/q}$. By the same reason as in Remark 3.2, we'll absorb $D_{p,q}$ into $\lesssim$. By a similar result of Lemma 3.3, we have

$$\|\sum_j |R_{d;j}|\|_{L^p_{x'}L^q_{x''}} \lesssim A_{d;p,q}, \quad \text{and} \quad \|\sum_j |\nabla R_{d;j}|\|_{L^p_{x'}L^q_{x''}} \lesssim B_{d;p,q},$$

where

$$A_{d;p,q} := \Big(\sum_j \omega_j^{\min(1,p,q)(\frac{1}{\alpha_1}-\frac{e}{2p}-\frac{d-e}{2q})}\Big)^{\max(1,p^{-1},q^{-1})},$$

$$B_{d;p,q} := \Big(\sum_j \langle v_j\rangle^{\min(1,p,q)} \omega_j^{\min(1,p,q)(\frac{1}{\alpha_1}-\frac{e}{2p}-\frac{d-e}{2q})}\Big)^{\max(1,p^{-1},q^{-1})}.$$

3. We need all the solitons in both sequences to be separated, hence we define

$$v_* = \min(v_*(T_e), v_*(T_d), v_*(T_e, T_d)),$$

where $v_*(T_e)$ and $v_*(T_d)$ are as defined by (3.7), and

$$v_*(T_e, T_d) := \inf_{j,k\in\mathbb{N}} \min(\sigma_k^{1/2}, \omega_j^{1/2})|u_k - v_j'|. \tag{4.14}$$

Here $v_j' = (v_{j,1}, \ldots, v_{j,e})$, the first e components of v_j. The convention that we add a coefficient $1/2$ in (3.7) but not in (4.14) is only to simplify some expressions.

4. We write $\mathcal{C}_A^{(d)}$ for the original $\mathcal{C}_A$, and $\mathcal{C}_A^{(e)}$ the e dimensional analogue. For the anisotropic case, we define

$$\mathcal{C}_A^{(e,d-e)} := \Big\{(p,q) \in (0,\infty] \times (0,\infty] : \frac{1}{\alpha_1} - \frac{e}{2p} - \frac{d-e}{2q} > 0\Big\}.$$

Similarly, if $\alpha_1 < 2$, we have $\mathcal{C}_B^{(d)}$, $\mathcal{C}_B^{(e)}$, and

$$\mathcal{C}_B^{(e,d-e)} := \Big\{(p,q) \in (0,\infty] \times (0,\infty] : \frac{1}{\alpha_1} - \frac{e}{2p} - \frac{d-e}{2q} > \frac{1}{2}\Big\}.$$

Lemma 3.6 can also be generalized. For example, if $\alpha_1 < 2$, $(q_1, q_2) \in \mathcal{C}_A^{(e,d-e)}$, $(p_1, p_2) \in \mathcal{C}_B^{(e,d-e)}$, then we can choose $\{\omega_j\}$ and $\{v_j\}$ so that $A_{d;q_1,q_2}$ and $B_{d;p_1,p_2}$ are as small as we like, and v_* as large as we like (see Appendix A). We shall not give a description of all the needed facts, but just claim that, as before, it suffices to check that all the indices of A, B appearing in our proofs lie in their corresponding controllable class $\mathcal{C}$.

To construct solutions of the form (4.13), as discussed in Section 1.3, we consider the operator Φ in (1.8) with source term $G + H$, where

$$G = f(T_e + \eta_e + T_d + \eta) - f(T_e + \eta_e + T_d),$$

$$H = f(T_e + \eta_e + T_d) - f(T_e + \eta_e) - \sum_j f(R_{d;j}).$$

For convenience, we further divide H into $H_1 + H_2$, where

$$H_1 = f(T_e + \eta_e + T_d) - f(T_e + \eta_e) - f(T_d),$$
$$H_2 = f(T_d) - \sum_j f(R_{d;j}).$$

The Strichartz estimate asserts

$$\|\Phi\eta\|_{S(t)} \lesssim \|G + H_1 + H_2\|_{N(t)}, \quad \text{and} \quad \|\nabla\Phi\eta\|_{S(t)} \lesssim \|\nabla G + \nabla H_1 + \nabla H_2\|_{N(t)}. \tag{4.15}$$

Estimates for H_2 (or ∇H_2) will be provided by Lemma 3.4.

We now give our first main result. Notice that "e" here corresponds to the role of "d" in Section 3.

Theorem 4.5. *Let $1 \leq e \leq 3$, $e < d \leq e+3$, and f satisfy Assumptions (F), (T)$_e$, and (T)$_d$. Suppose $2(\frac{1}{2} - \frac{1}{e}) < \alpha_1 \leq \alpha_2 \leq 4/d$. For fixed $0 < \rho, t_0 < \infty$, there is a constant $\lambda_0 > 0$ such that the following holds: For $\lambda_0 \leq \lambda < \infty$, there exist solutions of (1.1) of the form (4.13), with*

$$\sup_{t \geq t_0} e^{\lambda t} \left\{ \|\eta_e(t)\|_{L^2_{x'} \cap L^\infty_{x'}} + \|\eta\|_{S(t)} \right\} \leq \rho.$$

Remark. We need $\alpha_2 \leq 4/d$ so that we can bound $\||\eta|^{\alpha_2+1}\|_{N(t)}$ by $\|\eta\|_{S(t)}^{\alpha_2+1}$ from Lemma 4.4. We need $d \leq e+3$ in estimating $\||T_e + \eta_e||T_d|^{\alpha_1}\|_{N(t)}$. We may take $t_0 = 0$ if $\alpha_2 < 4/d$, or if ρ is sufficiently small.

Remark. It's most natural to view the eD-dD trains as solutions of (1.1) in $\mathbb{R}^d_x$, with dD solitons being "points" and eD solitons lower dimensional objects. Nevertheless, as we have mentioned in the introduction, we can also freely regard them as living in an even higher dimension, so that both e, d have nonzero codimensions to the ambient space.

Proof. We will only consider $\rho = 2$. The cases of other ρ can be treated similarly.

First, from the assumption, if $e = 2, 3$, then $d \geq 3$, and hence $\alpha_1 < 2$. Thus, for $e = 1, 2, 3$, if λ is large enough, Corollary 3.11 implies the existence of an eD train $T_e + \eta_e$ satisfying

$$\|\eta_e(t)\|_{L^2_{x'} \cap L^\infty_{x'}} \leq e^{-\lambda t}, \quad \forall t \geq t_0. \tag{4.16}$$

It remains to prove that Φ can be a contraction mapping on the closed unit ball of S_{λ,t_0}. As before, we'll only give estimates for Φ to be a self-mapping.

Suppose $\eta \in S_{\lambda,t_0}$ with $\|\eta\|_{S_{\lambda,t_0}} \leq 1$, i.e. $\|\eta\|_{S(t)} \leq e^{-\lambda t}$ for $t \geq t_0$. To estimate $\|\Phi\eta\|_{S_{\lambda,t_0}}$ from the Strichartz estimate (4.15), we have to estimate $\|G\|_{N(t)}$, $\|H_1\|_{N(t)}$ and $\|H_2\|_{N(t)}$. Since $\|\cdot\|_{N(t)} \leq \|\cdot\|_{L^1_t L^2_x}$, we'll frequently just estimate $\|\cdot\|_{L^1_t L^2_x}$. Also repeatedly used is the fact $\|\eta\|_{L^1_t L^2_x(t)} \leq \lambda^{-1} e^{-\lambda t}$, obtained from (4.2) (or Corollary 4.3).

Part 1. Estimate of $\|G\|_{N(t)}$. We have

$$|G| \lesssim \sum_{i=1,2} (|\eta||T_e + \eta_e + T_d|^{\alpha_i} + |\eta|^{\alpha_i+1}).$$

For the first term, we have

$$\begin{aligned}\| |\eta| |T_e+\eta_e+T_d|^{\alpha_i} \|_{L_t^1 L_x^2(t)} &\lesssim \|\eta\|_{L_t^1 L_x^2(t)} \|T_e+\eta_e+T_d\|^{\alpha_i}_{L_t^\infty L_x^\infty(t)} \\ &\lesssim (A_{e;\infty}+e^{-\lambda t_0}+A_{d;\infty})^{\alpha_i} \lambda^{-1} e^{-\lambda t},\end{aligned} \tag{4.17}$$

by (4.16). For the second term, since $\alpha_2 \le 4/d$ (and hence $\alpha_1 \le 4/d$), Lemma 4.4 (N0) implies

$$\| |\eta|^{\alpha_i+1} \|_{N(t)} = \| |\eta| |\eta|^{\alpha_i} \|_{N(t)} \lesssim (\lambda^{-1+d\alpha_i/4} e^{-\alpha_i \lambda t_0}) e^{-\lambda t}. \tag{4.18}$$

Notice that for the endpoint case $\alpha_i = 4/d$, the smallness of the coefficient (obtained by letting λ large) have to be provided by $e^{-\alpha_i \lambda t_0}$. This is the reason we consider an initial time $t_0 > 0$. By (4.17) and (4.18) we get the needed estimate of $\|G\|_{N(t)}$.

Part 2. Estimate of $\|H_1\|_{N(t)}$. By Corollary 2.5,

$$\begin{aligned}|H_1| &\lesssim \sum_{i=1,2} (|T_e+\eta_e|^{\max(1,\alpha_i)} |T_d|^{\min(1,\alpha_i)} + |T_e+\eta_e| |T_d|^{\alpha_i}) \\ &= |T_e+\eta_e| |T_d|^{\min(1,\alpha_1)} h_1,\end{aligned}$$

where

$$\|h_1\|_{L_t^\infty L_x^\infty(t)} \lesssim \sum_{i=1,2} A_{d;\infty}^{\min(1,\alpha_i)-\min(1,\alpha_1)} (A_{e;\infty}+e^{-\lambda t_0}+A_{d;\infty})^{\max(0,\alpha_i-1)}.$$

Thus it suffices to estimate $\| |\eta_e| |T_d|^{\min(1,\alpha_1)} \|_{N(t)}$ and $\| |T_e| |T_d|^{\min(1,\alpha_1)} \|_{N(t)}$. In the following we denote $\gamma = \min(1,\alpha_1)$ to save notation.

Part 2-1. Estimate of $\| |\eta_e| |T_d|^\gamma \|_{N(t)}$. Since $L_x^2 = L_{x'}^2 L_{x''}^2$,

$$\begin{aligned}\| |\eta_e| |T_d|^\gamma \|_{N(t)} &\le \| |\eta_e| |T_d|^\gamma \|_{L_t^1 L_{x'}^2 L_{x''}^2(t)} \\ &\le \|\eta_e\|_{L_t^1 L_{x'}^2 L_{x''}^\infty(t)} \| |T_d|^\gamma \|_{L_t^\infty L_{x'}^\infty L_{x''}^2(t)} \\ &\lesssim A_{d;\infty,2\gamma}^\gamma \lambda^{-1} e^{-\lambda t}.\end{aligned} \tag{4.19}$$

Now $(\infty, 2\gamma) \in \mathcal{C}_A^{(e,d-e)}$ means $\frac{1}{\alpha_1} > \frac{d-e}{4\gamma}$. If $\gamma = \alpha_1$, it's true since $d-e<4$. If $\gamma = 1$, it's true since $\alpha_1 \le 4/d$.

Part 2-2. Estimate of $\| |T_e| |T_d|^\gamma \|_{N(t)}$. We first prove the exponential decay of its L_x^2 norm by interpolation.

Step 1. For $s \in (0,\infty]$ and $\theta \in [0,1]$,

$$\| |T_e| |T_d|^\gamma \|_{L_x^s} \le \|T_e\|_{L_{x'}^{s/\theta} L_{x''}^\infty} \| |T_d|^\gamma \|_{L_{x'}^{s/(1-\theta)} L_{x''}^s} \lesssim A_{e;s/\theta} A_{d;\gamma s/(1-\theta),\gamma s}^\gamma. \tag{4.20}$$

We need $s/\theta \in \mathcal{C}_A^{(e)}$ and $(\gamma s/(1-\theta), \gamma s) \in \mathcal{C}_A^{(e,d-e)}$, that is

$$\frac{1}{\alpha_1} > \frac{e}{2(s/\theta)} \quad \text{and} \quad \frac{1}{\alpha_1} > \frac{e}{2(\gamma s/(1-\theta))} + \frac{d-e}{2\gamma s},$$

or equivalently

$$s > \max\left(\frac{e\theta\alpha_1}{2}, \frac{(d-e\theta)\alpha_1}{2\gamma} \right).$$

We hope this can be satisfied by some $s < 2$, by choosing a suitable θ. A little computation shows that the minimum of the "max" is achieved by letting

$$\theta = \min\left(\frac{d}{e(1+\gamma)}, 1\right). \tag{4.21}$$

Precisely we have the following alternatives:

1. If $\theta = \frac{d}{e(1+\gamma)} \leq 1$, we get $s > \frac{d\alpha_1}{2(1+\gamma)}$.
2. If $\theta = 1 < \frac{d}{e(1+\gamma)}$, we get $s > \frac{(d-e)\alpha_1}{2\gamma}$.

It's straightforward to check that, for all (e, d, α_1) satisfying our assumptions, the above lower bound of s is less than 2. (Here we use $d - e < 4$ again.) Thus, if θ is given by (4.21), there exists $0 < s_1 < 2$ such that (4.20) holds with $s = s_1$.

Step 2. We have

$$\begin{aligned}
\||T_e||T_d|^\gamma(\tau)\|_{L_x^\infty} &\leq \|(\sum_k |R_{e;k}|)(\sum_j |R_{d;j}|)^\gamma(\tau)\|_{L_x^\infty} \\
&\leq \|(\sum_k |R_{e;k}|)(\sum_j |R_{d;j}|^\gamma)(\tau)\|_{L_x^\infty} \quad (\text{since } \gamma \leq 1) \\
&= \|\sum_{k,j} |R_{e;k}||R_{d;j}|^\gamma(\tau)\|_{L_x^\infty} \\
&\leq \sum_{k,j} \||R_{e;k}||R_{d;j}|^\gamma(\tau)\|_{L_x^\infty}.
\end{aligned} \tag{4.22}$$

We also have

$$\begin{aligned}
|R_{e;k}||R_{d;j}|^\gamma(\tau) &\lesssim \sigma_k^{\frac{1}{\alpha_1}} \omega_j^{\frac{\gamma}{\alpha_1}} e^{-a\sigma_k^{1/2}|x'-u_k\tau| - a\gamma\omega_j^{1/2}|x-v_j\tau|} \\
&\leq \sigma_k^{\frac{1}{\alpha_1}} \omega_j^{\frac{\gamma}{\alpha_1}} e^{-a\sigma_k^{1/2}|x'-u_k\tau| - a\gamma\omega_j^{1/2}|x'-v_j'\tau|},
\end{aligned}$$

where recall that $v_j' = (v_{j,1}, \ldots, v_{j,e})$ consists of the first e components of v_j. Note that for any $c_1, c_2 > 0$ and $w_1, w_2 \in \mathbb{R}^n$,

$$b_1|x - w_1| + b_2|x - w_2| \geq \min(b_1, b_2)(|x - w_1| + |x - w_2|) \geq \min(b_1, b_2)|w_1 - w_2|.$$

Thus

$$\begin{aligned}
|R_{e;k}||R_{d;j}|^\gamma(\tau) &\lesssim \sigma_k^{\frac{1}{\alpha_1}} \omega_j^{\frac{\gamma}{\alpha_1}} e^{-a\gamma\min(\sigma_k^{1/2}, \omega_j^{1/2})|u_k - v_j'|\tau} \\
&\leq \sigma_k^{\frac{1}{\alpha_1}} \omega_j^{\frac{\gamma}{\alpha_1}} e^{-a\min(1,\alpha_1)v_*\tau}.
\end{aligned}$$

Taking this into (4.22), we get

$$\begin{aligned}
\||T_e||T_d|^\gamma(\tau)\|_{L_x^\infty} &\lesssim \sum_{k,j} \sigma_k^{\frac{1}{\alpha_1}} \omega_j^{\frac{\gamma}{\alpha_1}} e^{-a\min(1,\alpha_1)v_*\tau} \\
&= (\sum_k \sigma_k^{\frac{1}{\alpha_1}})(\sum_j \omega_j^{\frac{\gamma}{\alpha_1}}) e^{-a\min(1,\alpha_1)v_*\tau}.
\end{aligned} \tag{4.23}$$

The number $\sum_j \omega_j^{\frac{\gamma}{\alpha_1}}$ can be controlled as $A_{d;p}$ as described in Lemma 3.6. For preciseness, we can fix any $p_1 \in \mathcal{C}_A^{(d)}$ close to $d\alpha_1/2$ such that

$$\frac{\gamma}{\alpha_1} \geq \min(1, p_1)\Big(\frac{1}{\alpha_1} - \frac{d}{2p_1}\Big),$$

which implies

$$\sum_j \omega_j^{\frac{\gamma}{\alpha_1}} \lesssim \sum_j \omega_j^{\min(1,p_1)(\frac{1}{\alpha_1} - \frac{d}{2p_1})} = A_{d;p_1}^{\min(1,p_1)}.$$

Thus (4.23) gives

$$\| |T_e| |T_d|^\gamma(\tau)\|_{L_x^\infty} \lesssim A_{e;\infty} A_{d;p_1}^{\min(1,p_1)} e^{-a\min(1,\alpha_1)v_*\tau}. \tag{4.24}$$

From (4.20) (with θ given by (4.21) and $s = s_1 < 2$) and (4.24), we get

$$\| |T_e| |T_d|^\gamma(\tau)\|_{L_x^2} \lesssim E_2 e^{-(1-s_1/2)a\min(1,\alpha_1)v_*\tau}.$$

We omit the expression of E_2, which is obvious while cumbersome. Suppose

$$(1 - s_1/2)a\min(1, \alpha_1)v_* \geq \lambda, \tag{4.25}$$

we get

$$\| |T_e| |T_d|^\gamma\|_{N(t)} \leq \| |T_e| |T_d|^\gamma\|_{L_t^1 L_x^2(t)} \lesssim E_2 \lambda^{-1} e^{-\lambda t}.$$

Part 3. Estimate of $\|H_2\|_{N(t)}$. Choose $s_2 \in (\frac{d\alpha_1}{2(\alpha_1+1)}, 2)$ (it's easy to check that the interval is nonempty). Then Lemma 3.4 (H0) implies

$$\|H_2(\tau)\|_{L_x^2} \lesssim \Big(\sum_{i=1,2} A_{d;(\alpha_i+1)s_2}^{\alpha_i+1}\Big)^{s_2/2}\Big(\sum_{i=1,2} A_{d;\infty}^{\alpha_i+1}\Big)^{1-s_2/2} e^{-a(1-s_2/2)v_*\tau},$$

with $(\alpha_1 + 1)s_2 \in \mathcal{C}_A^{(d)}$. Thus, suppose

$$a(1 - s_2/2)v_* \geq \lambda, \tag{4.26}$$

we get

$$\|H_2\|_{N(t)} \leq \|H_2\|_{L_t^1 L_x^2(t)} \lesssim \Big(\sum_{i=1,2} A_{d;(\alpha_i+1)s_2}^{\alpha_i+1}\Big)^{s_2/2}\Big(\sum_{i=1,2} A_{d;\infty}^{\alpha_i+1}\Big)^{1-s_2/2} \lambda^{-1} e^{-\lambda t}.$$

From the conclusions in Part 1, Part 2, and Part 3, we are done. $\square$

Remark 4.6. Without using the anisotropic estimates for T_d, our assertions will be much weaker. For example, consider (4.19) in *Part 2-1*. If we do not use an anisotropic estimate of T_d, we can only estimate as follows: For any $(q, r) \in \mathcal{A}$

$$\begin{aligned} \| |\eta_e| |T_d|^\gamma\|_{N(t)} &\leq \| |\eta_e| |T_d|^\gamma\|_{L_t^{q'} L_x^{r'}(t)} \\ &\leq \|\eta_e\|_{L_t^{q'} L_x^\infty(t)} \| |T_d|^\gamma\|_{L_t^\infty L_x^{r'}(t)} \lesssim A_{d;\gamma r'}^\gamma \|\eta_e\|_{L_t^{q'} L_x^\infty(t)}. \end{aligned}$$

Now for $\gamma r' \in \mathcal{C}_A^{(d)}$, we need

$$\frac{d\alpha_1}{2\gamma} < r' \le 2. \tag{4.27}$$

If $d \ge 4$, we have $\gamma = \min(1, \alpha_1) = \alpha_1$ (since $\alpha_i \le 4/d$), and (4.27) is impossible. Thus only $d \le 3$ is allowed. Moreover, even for $d \le 3$, if $\gamma = 1$, the endpoint case $\alpha_1 = \alpha_2 = 4/d$ is excluded.

When $1 \le \alpha_1 < 2$, Theorem 3.9 implies the existence of an 1D train $T_1 + \eta_1$ such that $\|\eta_1(t)\|_{W_x^{1,\infty}}$ has exponential decay. This allows us to use the gradient estimate when $e = 1$. Precisely, we can try to construct a mixed train of the form $T_1 + \eta_1 + T_d + \eta$ ($d > 1$), by assuming the exponential decay of $\|\nabla\eta\|_{S(t)}$ (besides $\|\eta\|_{S(t)}$). It turns out that we can do it only for $d = 2$, and under a further restriction on α_1. The result is not only of its own interest, but also makes it possible to realize the 1D-2D-3D trains in the next section.

Theorem 4.7. *Let $e = 1$, $d = 2$, and f satisfy Assumptions (F), (T)$_1$, and (T)$_2$. Suppose moreover $1 \le \alpha_1 < 4/3$. Then for any finite $\rho > 0$, there is a constant $\lambda_0 > 0$ such that the following holds: For $\lambda_0 \le \lambda < \infty$, there exist solutions of* (1.1) *of the form* (4.13) *(namely $T_1 + \eta_1 + T_2 + \eta$) such that*

$$\sup_{t\ge 0} e^{\lambda t} \left\{ \|\eta_1(t)\|_{H^1_{x_1} \cap W^{1,\infty}_{x_1}} + \|\eta\|_{S(t)} + \|\nabla\eta\|_{S(t)} \right\} \le \rho.$$

Remark. We need $\alpha_1 < 4/3$ and $d = 2$ to bound $\|(T_1 + \eta_1)\nabla T_2\|_{N(t)}$. Note $t \ge t_0 = 0$ even for large ρ.

Proof. We will assume $\rho = 2$ for simplicity. For λ no less than some positive number, Theorem 3.9 implies the existence of an 1D train $T_1 + \eta_1$ satisfying

$$\|\eta_1(t)\|_{H^1_{x_1} \cap W^{1,\infty}_{x_1}} \le e^{-\lambda t}, \quad \forall t \ge 0.$$

In the following, we denote $S_{\lambda,0}$ (i.e. the initial time $t_0 = 0$) by S_λ, and let X be the Banach space of all $\eta : [0, \infty) \times \mathbb{R}^2 \to \mathbb{C}$ such that

$$\|\eta\|_X \coloneqq \|\eta\|_{S_\lambda} + \|\nabla\eta\|_{S_\lambda} < \infty.$$

We'll give estimates for Φ to be a self-mapping on the closed unit ball of X.

Suppose $\eta \in X$ with $\|\eta\|_X \le 1$. The estimate of $\|\Phi\eta\|_{S_\lambda}$ is the same as in the proof of Theorem 4.5, except for $\||\eta|^{\alpha_2+1}\|_{N(t)}$. Since the value of α_2 is not restricted, we use Lemma 4.4 (N1) instead of (N0) to obtain

$$\||\eta|^{\alpha_2+1}\|_{N(t)} \lesssim \lambda^{-\mu} e^{-(\alpha_2+1)\lambda t},$$

for some $\mu = \mu(d = 2, \alpha_2) > 0$. (For $\||\eta|^{\alpha_1+1}\|_{N(t)}$, both (N0) and (N1) work.) We remark that here and later we use μ as a generic constant, whose value may be different in different places.

Now we estimate $\|\nabla\Phi\eta\|_{S(t)}$. From (4.15), we have to estimate $\|\nabla G\|_{N(t)}$, $\|\nabla H_1\|_{N(t)}$, and $\|\nabla H_2\|_{N(t)}$.

Part 1. Estimate of $\|\nabla G\|_{N(t)}$. Let $W = T_1 + \eta_1 + T_2$, then $G = f(W+\eta) - f(W)$. Since $\alpha_1 \geq 1$, (2.4) implies

$$|\nabla G| \lesssim \sum_{i=1,2} \Big\{ |\eta|(|W| + |\eta|)^{\alpha_i - 1}|\nabla W| + (|W| + |\eta|)^{\alpha_i}|\nabla \eta| \Big\}.$$

Thus we have to estimate the $N(t)$ norm of (1) $|\eta||W|^{\alpha_i-1}|\nabla W|$, (2) $|\eta|^{\alpha_i}|\nabla W|$, (3) $|W|^{\alpha_i}|\nabla \eta|$, and (4) $|\eta|^{\alpha_i}|\nabla \eta|$. We discuss them in the following.

Estimate (1). We have

$$\begin{aligned} \||\eta||W|^{\alpha_i-1}|\nabla W|\|_{L_t^1 L_x^2(t)} &\leq \|W\|_{L_t^\infty L_x^\infty}^{\alpha_i - 1} \|\nabla W\|_{L_t^\infty L_x^\infty} \|\eta\|_{L_t^1 L_x^2} \\ &\lesssim (A_{1;\infty} + 1 + A_{2;\infty})^{\alpha_i - 1}(B_{1;\infty} + 1 + B_{2;\infty})\lambda^{-1}e^{-\lambda t}. \end{aligned}$$

Estimate (2). By Lemma 4.4 (N1),

$$\begin{aligned} \||\eta|^{\alpha_i}|\nabla W|\|_{N(t)} &\leq \|\nabla W\|_{L_t^\infty L_x^\infty} \||\eta|^{\alpha_i}\|_{N(t)} \\ &\lesssim (B_{1;\infty} + 1 + B_{2;\infty})\lambda^{-\mu}e^{-\alpha_i \lambda t}, \end{aligned}$$

for some $\mu > 0$.

Estimate (3).

$$\||W|^{\alpha_i}|\nabla \eta|\|_{L_t^1 L_x^2(t)} \leq \|W\|_{L_t^\infty L_x^\infty}^{\alpha_i} \|\nabla \eta\|_{L_t^1 L_x^2(t)} \lesssim (A_{1;\infty} + 1 + A_{2;\infty})^{\alpha_i}\lambda^{-1}e^{-\lambda t}.$$

Estimate (4). Also by Lemma 4.4 (N1) (with $u = |\nabla \eta|$ and $v = \eta$ there), we get

$$\||\eta|^{\alpha_i}|\nabla \eta|\|_{N(t)} \lesssim \lambda^{-\mu}e^{-(\alpha_i+1)\lambda t},$$

for some $\mu > 0$.

Part 2. Estimate of $\|\nabla H_1\|_{N(t)}$. Let $w = T_1 + \eta_1$. Since $\alpha_1 \geq 1$, (2.5) implies

$$\begin{aligned} |\nabla H_1| &= |\nabla[f(w + T_2) - f(w) - f(T_2)]| \\ &\lesssim \sum_{i=1,2} (|w| + |T_2|)^{\alpha_i - 1}(|w||\nabla T_2| + |T_2||\nabla w|). \end{aligned}$$

Since

$$\|(|w| + |T_2|)^{\alpha_i - 1}\|_{L_t^\infty L_x^\infty(t)} \lesssim (A_{1;\infty} + 1 + A_{2;\infty})^{\alpha_i - 1}, \tag{4.28}$$

it suffices to estimate (1) $|\eta_1||\nabla T_2|$, (2) $|T_2||\nabla \eta_1|$, (3) $|T_1||\nabla T_2|$, and (4) $|T_2||\nabla T_1|$. We discuss them in the following.

Estimate (1).

$$\begin{aligned} \||\eta_1||\nabla T_2|\|_{L_t^1 L_x^2(t)} &\leq \|\eta_1\|_{L_t^1 L_{x_1}^2 L_{x_2}^\infty(t)} \|\nabla T_2\|_{L_t^\infty L_{x_1}^\infty L_{x_2}^2(t)} \\ &\lesssim B_{2;\infty,2}\lambda^{-1}e^{-\lambda t}. \end{aligned}$$

We need $(\infty, 2) \in \mathcal{C}_B^{(1,1)}$, i.e. $\frac{1}{\alpha_1} - \frac{1}{2\cdot\infty} - \frac{1}{2\cdot 2} > \frac{1}{2}$. This is true by $\alpha_1 < 4/3$.

Notice that if we do not use an anisotropic estimate for ∇T_2, the requirement becomes $\alpha_1 < 1$, and the construction fails since we assume $\alpha_1 \geq 1$. Moreover, it is also due to this

part that the construction is valid only for $d = 2$. Indeed, suppose $d \geq 3$, with coordinates $x = (x_1, x'')$. If for some admissible (a', b'),

$$\| |\eta_1| |\nabla T_2| \|_{L_t^a L_x^b(t)} \leq \|\eta_1\|_{L_t^a L_{x_1}^2 L_{x''}^\infty(t)} \|\nabla T_2\|_{L_t^\infty L_{x_1}^p L_{x''}^b(t)},$$

where $1/p = 1/b - 1/2$ and $(p, b) \in \mathcal{C}_B^{(1,d-1)}$. It follows

$$\frac{1}{\alpha_1} > \frac{1}{2} + \frac{1}{2p} + \frac{d-1}{2b} > \frac{1}{2} + 0 + \frac{1}{b} \geq 1,$$

contradicting $1 \leq \alpha_1$.

Estimate (2).

$$\begin{aligned} \| |T_2| |\nabla \eta_1| \|_{L_t^1 L_x^2(t)} &\leq \|T_2\|_{L_t^\infty L_{x_1}^\infty L_{x_2}^2(t)} \|\nabla \eta_1\|_{L_t^1 L_{x_1}^2 L_{x_2}^\infty(t)} \\ &\lesssim A_{2;\infty,2} \lambda^{-1} e^{-\lambda t}, \end{aligned}$$

where $(\infty, 2) \in \mathcal{C}_A^{(1,1)}$.

Estimate (3). We will prove the exponential decay of $\|T_1 \nabla T_2\|_{L_x^2}$ by interpolation. First, for $s \in (0, \infty]$ and $\theta \in [0, 1]$,

$$\| |T_1| |\nabla T_2| \|_{L_x^s} \leq \|T_1\|_{L_{x_1}^{s/\theta} L_{x_2}^\infty} \|\nabla T_2\|_{L_{x_1}^{s/(1-\theta)} L_{x_2}^s} \lesssim A_{1;s/\theta} B_{2;s/(1-\theta),s}. \tag{4.29}$$

We need $s/\theta \in \mathcal{C}_A^{(1)}$ and $(s/(1-\theta), s) \in \mathcal{C}_B^{(1,1)}$, that is

$$\frac{1}{\alpha_1} - \frac{1}{2(s/\theta)} > 0, \quad \text{and} \quad \frac{1}{\alpha_1} - \frac{1}{2s/(1-\theta)} - \frac{1}{2s} > \frac{1}{2},$$

or equivalently

$$s > \max\left(\frac{\theta\alpha_1}{2}, \frac{(2-\theta)\alpha_1}{2-\alpha_1}\right).$$

The "max" is minimized by letting $\theta = 1$, which gives $s > \frac{\alpha_1}{2-\alpha_1}$. Since $\alpha_1 < 4/3$, the lower bound is less than 2. Thus (4.29) implies

$$\| |T_1| |\nabla T_2| \|_{L_x^{s_1}} \lesssim A_{1;s_1} B_{2;\infty,s_1} \tag{4.30}$$

for some $s_1 < 2$. Then consider the supremum estimate.

$$\begin{aligned} \| |T_1| |\nabla T_2| (\tau) \|_{L_x^\infty} &\leq \| (\sum_k |R_{1;k}|)(\sum_j |\nabla R_{2;j}|)(\tau) \|_{L_x^\infty} \\ &\leq \sum_{k,j} \| |R_{1;k}| |\nabla R_{2;j}| (\tau) \|_{L_x^\infty}. \end{aligned}$$

We have

$$\begin{aligned} |R_{1;k}| |\nabla R_{2;j}| (\tau) &\lesssim \sigma_k^{\frac{1}{\alpha_1}} \omega_j^{\frac{1}{\alpha_1}} \langle v_j \rangle e^{-a\sigma_k^{1/2}|x_1 - u_k\tau| - a\omega_j^{1/2}|x - v_j\tau|} \\ &\leq \sigma_k^{\frac{1}{\alpha_1}} \omega_j^{\frac{1}{\alpha_1}} \langle v_j \rangle e^{-a\sigma_k^{1/2}|x_1 - u_k\tau| - a\omega_j^{1/2}|x_1 - v_{j,1}\tau|} \\ &\leq \sigma_k^{\frac{1}{\alpha_1}} \omega_j^{\frac{1}{\alpha_1}} \langle v_j \rangle e^{-a\min(\sigma_k^{1/2}, \omega_j^{1/2})|u_k - v_{j,1}|\tau} \\ &\leq \sigma_k^{\frac{1}{\alpha_1}} \omega_j^{\frac{1}{\alpha_1}} \langle v_j \rangle e^{-av_*\tau}. \end{aligned}$$

Thus

$$\begin{aligned}
\||T_1||\nabla T_2|(\tau)\|_{L_x^\infty} &\lesssim \sum_{k,j} \sigma_k^{\frac{1}{\alpha_1}} \omega_j^{\frac{1}{\alpha_1}} \langle v_j \rangle e^{-av_*\tau} \\
&= (\sum_k \sigma_k^{\frac{1}{\alpha_1}})(\sum_j \omega_j^{\frac{1}{\alpha_1}} \langle v_j \rangle) e^{-av_*\tau} \\
&= A_{1;\infty} B_{2;\infty} e^{-av_*\tau}. \qquad (4.31)
\end{aligned}$$

From (4.30) and (4.31), we get

$$\||T_1||\nabla T_2|(\tau)\|_{L_x^2} \lesssim A_{1;s_1}^{s_1/2} B_{2;\infty,s_1}^{s_1/2} A_{1;\infty}^{1-s_1/2} B_{2;\infty}^{1-s_1/2} e^{-a(1-s_1/2)v_*\tau}.$$

Suppose

$$a(1-s_1/2)v_* \geq \lambda, \qquad (4.32)$$

then we get

$$\||T_1||\nabla T_2|\|_{N(t)} \leq A_{1;s_1}^{s_1/2} B_{2;\infty,s_1}^{s_1/2} A_{1;\infty}^{1-s_1/2} B_{2;\infty}^{1-s_1/2} \lambda^{-1} e^{-\lambda t}.$$

Estimate (4). The strategy is the same. For $s > 0$ and $\theta \in [0,1]$,

$$\||T_2||\nabla T_1|\|_{L_x^s} \leq \|T_2\|_{L_{x_1}^{s/(1-\theta)} L_{x_2}^s} \|\nabla T_1\|_{L_{x_1}^{s/\theta} L_{x_2}^\infty} \lesssim A_{2;s/(1-\theta),s} B_{1;s/\theta}.$$

For $(s/(1-\theta), s) \in \mathcal{C}_A^{(1,1)}$ and $s/\theta \in \mathcal{C}_B^{(1)}$, we need

$$s > \max\left(\frac{(2-\theta)\alpha_1}{2}, \frac{\theta\alpha_1}{2-\alpha_1}\right).$$

The "max" is minimized by letting $\theta = \frac{2(2-\alpha_1)}{4-\alpha_1}$, which gives $s > \frac{2\alpha_1}{4-\alpha_1}$, where the lower bound is less than 2. Hence

$$\||T_2||\nabla T_1|\|_{L_x^{s_2}} \lesssim A_{2;s_2(4-\alpha_1)/\alpha_1,s_2} B_{1;s_2(4-\alpha_1)/(4-2\alpha_1)}$$

for some $s_2 < 2$. Next,

$$\begin{aligned}
\||T_2||\nabla T_1|(\tau)\|_{L_x^\infty} &\leq \sum_{k,j} \||R_{2;j}||\nabla R_{1;k}|(\tau)\|_{L_x^\infty} \\
&\lesssim \sum_{k,j} \omega_j^{\frac{1}{\alpha_1}} \sigma_k^{\frac{1}{\alpha_1}} \langle u_k \rangle e^{-av_*\tau} \\
&\lesssim A_{2;\infty} B_{1;\infty} e^{-av_*\tau}.
\end{aligned}$$

By interpolation we get

$$\||T_2||\nabla T_1|(\tau)\|_{L_x^2} \lesssim A_{2;s_2(4-\alpha_1)/\alpha_1,s_2}^{s_2/2} B_{1;s_2(4-\alpha_1)/(4-2\alpha_1)}^{s_2/2} A_{2;\infty}^{1-s_2/2} B_{1;\infty}^{1-s_2/2} e^{-a(1-s_2/2)v_*\tau}.$$

Suppose

$$a(1-s_2/2)v_* \geq \lambda, \qquad (4.33)$$

then we get

$$\||T_2||\nabla T_2|\|_{N(t)} \lesssim A_{2;s_2(4-\alpha_1)/\alpha_1,s_2}^{s_2/2} B_{1;s_2(4-\alpha_1)/(4-2\alpha_1)}^{s_2/2} A_{2;\infty}^{1-s_2/2} B_{1;\infty}^{1-s_2/2} \lambda^{-1} e^{-\lambda t}.$$

Part 3. Estimate of $\|\nabla H_2\|_{N(t)}$. Choose $2 > s_3 > \frac{2\alpha_1}{\alpha_1+2}$. By Lemma 3.4 (H1), we get

$$\|\nabla H_2(\tau)\|_{L_x^2} \lesssim (\sum_{i=1,2} A_{2;\alpha_i q}^{\alpha_i} B_{2;p})^{s_3/2} (\sum_{i=1,2} A_{2;\infty}^{\alpha_i} B_{2;\infty})^{1-s_3/2} e^{-a\min(\alpha_1,1)(1-s_3/2)v_*\tau},$$

where p, q are arbitrary numbers in $(0,\infty]$ satisfying $\frac{1}{q}+\frac{1}{p}=\frac{1}{s_3}$. Since

$$\frac{1}{s_3} < \frac{1}{\alpha_1} + \frac{1}{2} = 1 + (\frac{1}{\alpha_1} - \frac{1}{2}),$$

we can choose p, q such that $\frac{1}{q} < 1$ and $\frac{1}{p} < \frac{1}{\alpha_1} - \frac{1}{2}$. Thus $\alpha_1 q \in \mathcal{C}_A^{(2)}$ and $p \in \mathcal{C}_B^{(2)}$. Suppose

$$a\min(\alpha_1,1)(1-s_3/2)v_* \geq \lambda, \tag{4.34}$$

then we get

$$\|\nabla H_2\|_{N(t)} \lesssim (\sum_{i=1,2} A_{2;\alpha_i q}^{\alpha_i} B_{2;p})^{s_3/2} (\sum_{i=1,2} A_{2;\infty}^{\alpha_i} B_{2;\infty})^{1-s_3/2} \lambda^{-1} e^{-\lambda t}.$$

Combining all three parts, we get

$$\|\nabla \Phi \eta\|_{S(t)} \lesssim \|\nabla G\|_{N(t)} + \|\nabla H_1\|_{N(t)} + \|\nabla H_2\|_{N(t)} \lesssim \lambda^{-\mu} e^{-\lambda t}$$

for some $\mu > 0$. □

4.3 Construction of 1D-2D-3D trains

In this subsection, as our last main result, we construct 1D-2D-3D trains of the form

$$u = T_1 + \eta_1 + T_2 + \eta_2 + T_3 + \eta, \tag{4.35}$$

where $T_1 = T_1(t, x_1)$, $T_2 = T_2(t, x_1, x_2)$, and $T_3 = T_3(t, x)$ ($x = (x_1, x_2, x_3)$) are 1D, 2D, and 3D soliton train profiles respectively, with initial positions of all the solitons being the origin. $\eta_1 = \eta_1(t, x_1)$ and $\eta_2 = \eta_2(t, x_1, x_2)$ is such that $T_1 + \eta_1 + T_2 + \eta_2$ is an 1D-2D mixed train (the fact that $T_1 + \eta_1$ is itself an 1D train will not be explicitly needed later), and $\eta = \eta(t, x)$ is the remaining error to be found. To be precise, let $T_1 = \sum_k R_{1;k}$, where $R_{1;k}$ have frequencies $\sigma_{1;k}$ and velocities $(u_{1;k}, 0, 0)$; $T_2 = \sum_k R_{2;k}$, where $R_{2;k}$ have frequencies $\sigma_{2;k}$ and velocities $(u_{2;k,1}, u_{2;k,2}, 0)$; and $T_3 = \sum_j R_{3;j}$, where $R_{3;j}$ has frequencies ω_j and velocities $(v_{j,1}, v_{j,2}, v_{j,3})$. And we define

$$v_* = \min(v_*(T_1), v_*(T_2), v_*(T_3), v_*(T_1,T_2), v_*(T_1,T_3), v_*(T_2,T_3)), \tag{4.36}$$

where the numbers in the min are defined by (3.7) and (4.14).

(4.35) can be visualized as a plane-line-point soliton train in 3D space. It turns out to be the only mixed trains involving more than two dimensions that we can construct. To see this, we first give a discussion on the control of lower dimensional errors.

As we stressed, supremum controls in x for lower dimensional objects are necessary in constructing mixed trains. For the previous theorems on eD-dD trains, we use controls of the form

$$\|\eta_e(t)\|_{L_x^\infty} \le e^{-\lambda t} \tag{4.37}$$

established in Section 3. In fact, it is also possible to use space-time controls of the form

$$\|\eta_e\|_{L_t^p L_x^\infty(t)} \le e^{-\lambda t}, \tag{4.38}$$

for suitable p. In 1D space, since $(4,\infty) \in \mathcal{A}^{(1)}$, we can obtain $\|\eta_1\|_{L_t^4 L_x^\infty(t)}$ control by constructing $T_1 + \eta_1$ such that $\|\eta_1\|_{S(t)}$ has exponential decay in t. For $e = 2, 3$, since $r_{\max}^{(e)} > e$ (recall (4.1)), (4.38) can be obtained from the exponential decay of $\|\nabla\eta_e\|_{S(t)}$ and some $\|\eta_e\|_{L_t^q L_x^2(t)}$ (e.g. (1.10)) by Gagliardo-Nirenberg's inequality. For $e \ge 4$, (4.38) is not available (unless controls of even higher order derivatives of η_e are considered, which we did not pursue).

There is actually no definite reason we followed a route of using (4.37) but not (4.38) in constructing eD-dD trains. As to mixed trains involving more than two dimensions, all the lower dimensional errors have to have spatial supremum controls. As a consequence, thanks to Theorem 4.7, one sees that (4.35) becomes the only possible case, where we have type (4.37) control of η_1 and type (4.38) control of η_2. The details will be given in the proof of Theorem 4.8.

Since $T_1 + \eta_1 + T_2 + \eta_2$ is assumed to be a solution, the source term of Φ with respect to (4.35) becomes

$$f(T_1 + \eta_1 + T_2 + \eta_2 + T_3 + \eta) - f(T_1 + \eta_1 + T_2 + \eta_2) - \sum_j f(R_{3;j}).$$

We will write it as $G + H_1 + H_2$, where

$$\begin{aligned} G &= f(T_1 + \eta_1 + T_2 + \eta_2 + T_3 + \eta) - f(T_1 + \eta_1 + T_2 + \eta_2 + T_3), \\ H_1 &= f(T_1 + \eta_1 + T_2 + \eta_2 + T_3) - f(T_1 + \eta_1 + T_2 + \eta_2) - f(T_3), \\ H_2 &= f(T_3) - \sum_j f(R_{3;j}). \end{aligned}$$

Our main theorem is the following.

Theorem 4.8. *Let $d = 3$, and f satisfy Assumptions (F), (T)$_1$, (T)$_2$, and (T)$_3$. Suppose $1 \le \alpha_1 < 4/3$ and $\alpha_1 \le \alpha_2 \le 4/3$. For any finite $\rho, t_0 > 0$, there is a constant $\lambda_0 > 0$ such that the following holds: For $\lambda_0 \le \lambda < \infty$, there exist solutions of (1.1) of the form (4.35), such that*

$$\sup_{t \ge t_0} e^{\lambda t} \left\{ \|\eta_1(t)\|_{H_{x_1}^1 \cap W_{x_1}^{1,\infty}} + \|\eta_2\|_{S(t)} + \|\nabla\eta_2\|_{S(t)} + \|\eta\|_{S(t)} \right\} \le \rho. \tag{4.39}$$

Remark. We can take $t_0 = 0$ if $\alpha_2 < 4/3$, or if ρ is sufficiently small. The highest dimension cannot be larger than 3 in order to estimate terms of the form $\||\eta_1 + \eta_2|^\beta |T_3|^\gamma\|_{N(t)}$ for $\gamma = 1$ and α_1 ($\beta > 0$ is irrelevant).

Proof. For λ no less than some positive number, Theorem 4.7 implies the existence of an 1D-2D train $T_1+\eta_1+T_2+\eta_2$ such that

$$\|\eta_1(t)\|_{H^1_{x_1}\cap W^{1,\infty}_{x_1}} \le e^{-\lambda t}, \tag{4.40}$$

and

$$\|\eta_2\|_{S(t)} + \|\nabla\eta_2\|_{S(t)} \le e^{-\lambda t}. \tag{4.41}$$

We'll not exploit gradient estimates in this proof, and hence we don't need the control of $\nabla\eta_1$. The control for $\nabla\eta_2$ is needed merely to induce a type (4.38) control of η_2, as we show now. Denote $x'=(x_1,x_2)$. From (4.41), we have $\|\eta_2\|_{L^\infty_t L^2_{x'}(t)} \le e^{-\lambda t}$ and $\|\nabla\eta_2\|_{L^4_t L^4_{x'}} \le e^{-\lambda t}$. By the Gagliardo-Nirenberg's inequality (cf. (3.31)), for $0<p\le\infty$,

$$\begin{aligned}
\|\eta_2\|_{L^p_t L^\infty_{x'}} &\lesssim \left\| \|\eta_2\|^{1/3}_{L^2_{x'}} \|\nabla\eta_2\|^{2/3}_{L^4_{x'}} \right\|_{L^p_t} \\
&\le \left\| \|\eta_2\|^{1/3}_{L^2_{x'}} \right\|_{L^\infty_t} \left\| \|\nabla\eta_2\|^{2/3}_{L^4_{x'}} \right\|_{L^p_t} \\
&= \|\eta_2\|^{1/3}_{L^\infty_t L^2_{x'}} \|\nabla\eta_2\|^{2/3}_{L^{2p/3}_t L^4_{x'}}.
\end{aligned}$$

Letting $p=6$, we get

$$\|\eta_2\|_{L^6_t L^\infty_{x'}(t)} \lesssim e^{-\lambda t}. \tag{4.42}$$

Suppose $\eta\in S_{\lambda,t_0}$, $\|\eta\|_{S_{\lambda,t_0}}\le 1$. We will derive the suitable estimates of $\|G\|_{N(t)}$, $\|H_1\|_{N(t)}$ and $\|H_2\|_{N(t)}$ for $\|\Phi\eta\|_{S_{\lambda,t_0}}\le 1$ in the following.

Part 1. Estimate of $\|G\|_{N(t)}$. By (2.1),

$$|G| \lesssim \sum_{i=1,2} \left\{ |\eta||T_1+\eta_1+T_2+T_3|^{\alpha_i} + |\eta||\eta_2|^{\alpha_i} + |\eta|^{\alpha_i+1} \right\}.$$

For the first term, by (4.40),

$$\begin{aligned}
\||\eta||T_1+\eta_1+T_2+T_3|^{\alpha_i}\|_{L^1_t L^2_x(t)} &\le \|\eta\|_{L^1_t L^2_x(t)} \|T_1+\eta_1+T_2+T_3\|^{\alpha_i}_{L^\infty_t L^\infty_x} \\
&\lesssim (A_{1;\infty}+e^{-\lambda t_0}+A_{2;\infty}+A_{3;\infty})^{\alpha_i}\lambda^{-1}e^{-\lambda t}.
\end{aligned}$$

For the second, by (4.42), we have

$$\begin{aligned}
\||\eta||\eta_2|^{\alpha_i}\|_{L^1_t L^2_x(t)} &\le \|\eta\|_{L^{6/(6-\alpha_i)}_t L^2_x(t)} \||\eta_2|^{\alpha_i}\|_{L^{6/\alpha_i}_t L^\infty_x(t)} \\
&\lesssim (\lambda^{-(6-\alpha_i)/6}e^{-\alpha_i\lambda t_0})e^{-\lambda t}.
\end{aligned}$$

For the third, since $\alpha_2\le 4/3$, Lemma 4.4 (N0) implies

$$\||\eta|^{\alpha_i+1}\|_{N(t)} \lesssim (\lambda^{-1+3\alpha_i/4}e^{-\alpha_i\lambda t_0})e^{-\lambda t}.$$

(If $\alpha_2=4/3$, for $i=2$ the smallness of the coefficient by letting λ large relies on the assumption $t_0>0$.)

Part 2. Estimate of $\|H_1\|_{N(t)}$. Let $W=T_1+\eta_1+T_2+\eta_2$. By Corollary 2.5,

$$|H_1| = |f(W+T_3)-f(W)-f(T_3)| \lesssim \sum_{i=1,2}(|W|^{\alpha_i}|T_3|+|W||T_3|^{\alpha_i}).$$

Thus we have to estimate (1) $|\eta_1|^\beta |T_3|^\gamma$, (2) $|\eta_2|^\beta |T_3|^\gamma$, (3) $|T_1|^\beta |T_3|^\gamma$, and (4) $|T_2|^\beta |T_3|^\gamma$, for $(\beta,\gamma) = (\alpha_i, 1)$ and $(1, \alpha_i)$. Notice that by assumption both $\beta, \gamma \ge 1$ in any case.

Estimate (1).

$$\||\eta_1|^\beta |T_3|^\gamma\|_{L^1_t L^2_x(t)} \le \||\eta_1|^\beta\|_{L^1_t L^\infty_x(t)} \||T_3|^\gamma\|_{L^\infty_t L^2_x(t)} \lesssim A^\gamma_{3;2\gamma} \|\eta_1\|^\beta_{L^\beta_t L^\infty_x(t)},$$

where $2\gamma \in \mathcal{C}^{(3)}_A$ since $\gamma \ge 1$ and $\alpha_1 < 4/3$. (Notice that, for $\gamma = 1$ and α_1, $2\gamma \notin \mathcal{C}^{(d)}_A$ if $d \ge 4$. This is why we can only consider 3 as the highest dimension.) By (4.40),

$$\|\eta_1\|^\beta_{L^\beta_t L^\infty_x(t)} \le \int_t^\infty e^{-\beta\lambda\tau}\, d\tau = (\beta\lambda)^{-1} e^{-\beta\lambda t}.$$

Hence

$$\||\eta_1|^\beta |T_3|^\gamma\|_{L^1_t L^2_x(t)} \lesssim (A^\gamma_{3;2\gamma} \lambda^{-1} e^{-(\beta-1)\lambda t_0}) e^{-\lambda t}.$$

Estimate (2). As above we get

$$\||\eta_2|^\beta |T_3|^\gamma\|_{L^1_t L^2_x(t)} \lesssim A^\gamma_{3;2\gamma} \|\eta_2\|^\beta_{L^\beta_t L^\infty_x}.$$

Let $u(\tau) = \|\eta_2(\tau)\|_{L^\infty_x}$. (4.42) says $\|u\|_{L^6([t,\infty))} \lesssim e^{-\lambda t}$. Since $\beta < 6$, Proposition 4.1 implies

$$\|\eta_2\|_{L^\beta_t L^\infty_x(t)} = \|u\|_{L^\beta([t,\infty))} \lesssim \lambda^{\frac{1}{6} - \frac{1}{\beta}} e^{-\lambda t}.$$

Thus

$$\||\eta_2|^\beta |T_3|^\gamma\|_{L^1_t L^2_x(t)} \lesssim (A^\gamma_{3;2\gamma} \lambda^{-1+\beta/6} e^{-(\beta-1)\lambda t_0}) e^{-\lambda t}.$$

Estimate (3). Since

$$|T_1|^{\alpha_i} |T_3| + |T_1||T_3|^{\alpha_i} \lesssim |T_1||T_3|(|T_1| + |T_3|)^{\alpha_i - 1}$$

and

$$\|(|T_1| + |T_3|)^{\alpha_i - 1}\|_{L^\infty_t L^\infty_x(t)} \lesssim (A_{1;\infty} + A_{3;\infty})^{\alpha_i - 1},$$

it suffices to estimate $\||T_1||T_3|\|_{N(t)}$.

Step 1. For $s \in (0, \infty]$ and $\theta \in [0, 1]$, with $x'' = (x_2, x_3)$,

$$\||T_1||T_3|\|_{L^s_x} \le \|T_1\|_{L^{s/\theta}_{x_1} L^\infty_{x''}} \|T_3\|_{L^{s/(1-\theta)}_{x_1} L^s_{x''}} \lesssim A_{1;s/\theta} A_{3;s/(1-\theta),s}. \tag{4.43}$$

Here $A_{3;s/(1-\theta),s}$ is with respect to $(e, d) = (1, 3)$. We need $s/\theta \in \mathcal{C}^{(1)}_A$ and $(s/(1-\theta), s) \in \mathcal{C}^{(1,2)}_A$, that is

$$s > \max\left(\frac{\theta\alpha_1}{2}, \frac{(3-\theta)\alpha_1}{2}\right).$$

The "max" is minimized by letting $\theta = 1$, which gives $s > \alpha_1$. Since $\alpha_1 < 2$, (4.43) gives

$$\||T_1||T_3|\|_{L^{s_1}_x} \lesssim A_{1;s_1} A_{3;\infty,s_1} \tag{4.44}$$

for some $s_1 < 2$, with $s_1 \in \mathcal{C}^{(1)}_A$ and $(\infty, s_1) \in \mathcal{C}^{(1,2)}_A$.

Step 2. Following the derivation of (4.24), we get

$$\||T_1||T_3|(\tau)\|_{L^\infty_x} \le \sum_{k,j} |R_{1;k}||R_{3;j}|(\tau) \lesssim A_{1;\infty}A_{3;\infty}e^{-av_*\tau}. \tag{4.45}$$

From (4.44) and (4.45), we get

$$\||T_1||T_3|(\tau)\|_{L^2_x} \lesssim A_{1;s_1}^{s_1/2}A_{3;\infty,s_1}^{s_1/2}A_{1;\infty}^{1-s_1/2}A_{3;\infty}^{1-s_1/2}e^{-a(1-s_1/2)v_*\tau}.$$

Suppose

$$a(1-s_1/2)v_* \ge \lambda, \tag{4.46}$$

we get

$$\||T_1||T_3|\|_{N(t)} \lesssim A_{1;s_1}^{s_1/2}A_{3;\infty,s_1}^{s_1/2}A_{1;\infty}^{1-s_1/2}A_{3;\infty}^{1-s_1/2}\lambda^{-1}e^{-\lambda t}.$$

Estimate (4). As above, it suffices to estimate $\||T_2||T_3|\|_{N(t)}$. Let $x' = (x_1, x_2)$. For $s \in (0,\infty]$ and $\theta \in [0,1]$,

$$\||T_2||T_3|\|_{L^s_x} \le \|T_2\|_{L^{s/\theta}_{x'}L^\infty_{x_3}}\|T_3\|_{L^{s/(1-\theta)}_{x'}L^s_{x_3}} \lesssim A_{2;s/\theta}A_{3;s/(1-\theta),s}. \tag{4.47}$$

Here $A_{3;s/(1-\theta),s}$ is with respect to $(e,d) = (2,3)$. For $s/\theta \in \mathcal{C}^{(2)}_A$ and $(s/(1-\theta), s) \in \mathcal{C}^{(2,1)}_A$, we need

$$s > \max\left(\alpha_1\theta, (\frac{3}{2}-\theta)\alpha_1\right).$$

The "max" is minimized by letting $\theta = 3/4$, which gives $s > \frac{3\alpha_1}{4}$. The lower bound is less than 2. Thus there is $s_2 < 2$ such that

$$\||T_2||T_3|\|_{L^{s_2}_x} \lesssim A_{2;4s_2/3}A_{3;4s_2,s_2},$$

with $4s_2/3 \in \mathcal{C}^{(2)}_A$ and $(4s_2, s_2) \in \mathcal{C}^{(2,1)}_A$.

By (4.24), we have

$$\||T_2||T_3|(\tau)\|_{L^\infty_x} \lesssim A_{2;\infty}A_{3;\infty}e^{-av_*\tau}.$$

By interpolation we get the L^2_x estimate. And if $a(1-s_2/2)v_* \ge \lambda$, we get

$$\||T_2||T_3|\|_{N(t)} \lesssim A_{2;4s_2/3}^{s_2/2}A_{3;4s_2,s_2}^{s_2/2}A_{2;\infty}^{1-s_2/2}A_{3;\infty}^{1-s_2/2}\lambda^{-1}e^{-\lambda t}.$$

Part 3. Estimate of $\|H_2\|_{N(t)}$. Choose $2 > s_3 > \frac{3\alpha_1}{2(\alpha_1+1)}$. By Lemma 3.4 (H0),

$$\|H_2(\tau)\|_{L^2_x} \lesssim (\sum_{i=1,2} A_{3;(\alpha_i+1)s_3}^{\alpha_i+1})^{s_3/2}(\sum_{i=1,2} A_{3;\infty}^{\alpha_i+1})^{1-s_3/2}e^{-a(1-s_3/2)v_*\tau},$$

with $(\alpha_1+1)s_3 \in \mathcal{C}^{(3)}_A$. Suppose $a(1-s_3/2)v_* \ge \lambda$, we get

$$\|H_2\|_{L^1_tL^2_x(t)} \lesssim (\sum_{i=1,2} A_{3;(\alpha_i+1)s_3}^{\alpha_i+1})^{s_3/2}(\sum_{i=1,2} A_{3;\infty}^{\alpha_i+1})^{1-s_3/2}\lambda^{-1}e^{-\lambda t}.$$

Combining all three parts, we see $\|\Phi\eta\|_{S(t)} \le e^{-\lambda t}$ for λ large enough with suitable frequencies and velocities of the solitons.

□

Appendix A

We prove Lemma 3.6 in this appendix. We will consider slightly more general forms of the assertions, so that they actually cover the anisotropic cases used in Section 4.

For $p_1, p_2 \in (0, \infty)$, sequence $\{\omega_j\}_{j\in\mathbb{N}}$ in $(0, \omega_*)$, and sequence $\{v_j\}_{j\in\mathbb{N}}$ in $\mathbb{R}^d$, define

$$\widetilde{A}_{p_1,p_2} = \widetilde{A}_{p_1,p_2}(\{\omega_j\}) := (\sum_j \omega_j^{p_1 p_2})^{1/p_1}$$
$$\widetilde{B}_{p_1,p_2} = \widetilde{B}_{p_1,p_2}(\{\omega_j\}, \{v_j\}) := (\sum_j \langle v_j\rangle^{p_1} \omega_j^{p_1 p_2})^{1/p_1}.$$

Proposition. *Given* $0 < p_1 \le q_1 < \infty$ *and* $0 < p_2 < q_2 < \infty$. *We have*

$$\begin{aligned} \widetilde{A}_{q_1,q_2} &< \omega_*^{q_2-p_2} \widetilde{A}_{p_1,p_2}, \quad \textit{if} \quad \widetilde{A}_{q_1,q_2} < \infty, \\ \widetilde{B}_{q_1,q_2} &< \omega_*^{q_2-p_2} \widetilde{B}_{p_1,p_2}, \quad \textit{if} \quad \widetilde{B}_{q_1,q_2} < \infty. \end{aligned}$$

Remark. By letting $p_1 = \min(1, p)$, $p_2 = \frac{1}{\alpha_1} - \frac{d}{2p}$, $q_1 = \min(1, q)$, $q_2 = \frac{1}{\alpha_1} - \frac{d}{2q}$, and notice that

$$\omega_*^{q_2-p_2} \le \max(1, \omega_*)^{q_2-p_2} \le \max(1, \omega_*)^{q_2} \le \max(1, \omega_*)^{1/\alpha_1},$$

we get Lemma 3.6 (a).

Proof. We have

$$\begin{aligned} \widetilde{A}_{q_1,q_2} &= \Big[(\sum_j \omega_j^{q_1 q_2})^{p_1/q_1} \Big]^{1/p_1} \le (\sum_j \omega_j^{p_1 q_2})^{1/p_1} \quad (\text{since } p_1/q_1 \le 1) \\ &= \omega_*^{q_2} (\sum_j (\omega_j/\omega_*)^{p_1 q_2})^{1/p_1} < \omega_*^{q_2} (\sum_j (\omega_j/\omega_*)^{p_1 p_2})^{1/p_1} \quad (\text{since } \omega_j/\omega_* < 1) \\ &= \omega_*^{q_2-p_2} \widetilde{A}_{p_1,p_2}. \end{aligned}$$

Similarly,

$$\widetilde{B}_{q_1,q_2} \le (\sum_j \langle v_j\rangle^{p_1} \omega_j^{p_1 q_2})^{1/p_1} < \omega_*^{q_2} (\sum_j \langle v_j\rangle^{p_1} (\omega_j/\omega_*)^{p_1 p_2})^{1/p_1} = \omega_*^{q_2-p_2} \widetilde{B}_{q_1,q_2}.$$

□

Let v_* be as defined by (3.7). Lemma 3.6 (b) is a corollary of the following

Proposition. *Given* $0 < p_1, q_1, q_2 < \infty$ *and* $1/2 < p_2 < \infty$. *For any constants* $c, \Lambda > 0$, *there exist* $\{\omega_j\}$ *and* $\{v_j\}$ *such that* $\widetilde{A}_{q_1,q_2}, \widetilde{B}_{p_1,p_2} \le c$ *and* $v_* \ge \Lambda$.

Proof. For constants $0 < \rho < 1$, $\gamma > 0$, and $\delta \ge 0$, let $\omega_j = \omega_* \rho^{2j}$, and v_j satisfies

$$|v_j| = \gamma(\sum_{\ell=2}^{j} \rho^{-\ell}) + \delta.$$

(The empty summation $\sum_{\ell=2}^{1}$ is understood to be zero.) Then for $j<k$ we have

$$\min(\omega_j^{1/2},\omega_k^{1/2})|v_j-v_k| \geq \omega_k^{1/2}(|v_k|-|v_j|) = \omega_*^{1/2}\rho^k\cdot\gamma(\sum_{\ell=j+1}^{k}\rho^{-\ell}).$$

Since $\rho^k(\sum_{\ell=j+1}^{k}\rho^{-\ell})>1$ ($\forall\rho\in(0,1)$), $v_*\geq\Lambda$ as long as

$$\gamma\geq 2\omega_*^{-1/2}\Lambda.$$

To complete the proof, it suffices to show that $\widetilde{A}_{q_1,q_2},\widetilde{B}_{p_1,p_2}\to 0$ as $\rho\to 0$. For $\widetilde{A}_{q_1,q_2}$, we have

$$\lim_{\rho\to 0}\widetilde{A}_{q_1,q_2}^{q_1} = \omega_*^{q_1q_2}\lim_{\rho\to 0}\sum_j \rho^{2q_1q_2j} = 0.$$

On the other hand, since $\langle v_j\rangle \lesssim |v_j|+1 = \gamma(\sum_{\ell=2}^{j}\rho^{-\ell})+(\delta+1)$,

$$\begin{aligned}\widetilde{B}_{p_1,p_2}^{p_1} &\lesssim \sum_j\Big[\gamma^{p_1}(\sum_{\ell=2}^{j}\rho^{-\ell})^{p_1}+(\delta+1)^{p_1}\Big]\omega_*^{p_1p_2}\rho^{2p_1p_2j}\\ &= \gamma^{p_1}\omega_*^{p_1p_2}I(\rho)+(\delta+1)^{p_1}\widetilde{A}_{p_1,p_2}^{p_1},\end{aligned}$$

where

$$\begin{aligned}I(\rho) &= \sum_j(\sum_{\ell=2}^{j}\rho^{-\ell})^{p_1}\omega^{2p_1p_2j} = \sum_j(\sum_{\ell=2}^{j}\rho^{-\ell})^{p_1}\rho^{p_1j}\cdot\rho^{-p_1j}\omega^{2p_1p_2j}\\ &= \sum_j(\sum_{\ell=2}^{j}\rho^{j-\ell})^{p_1}\rho^{2p_1(p_2-1/2)j} \leq \sum_j(\sum_{\ell=0}^{\infty}\rho^{\ell})^{p_1}\rho^{2p_1(p_2-1/2)j}\\ &= (1-\rho)^{-p_1}\cdot\frac{\rho^{2p_1(p_2-1/2)}}{1-\rho^{2p_1(p_2-1/2)}}\to 0 \quad\text{as}\quad \rho\to 0.\end{aligned}$$

$\square$

Appendix B

Let $x=(x',x'')$ be as in Section 4.2. One would wonder if the $L_{x'}^pL_{x''}^q$ norm can be bounded by the $L_x^p\cap L_x^q$ norm. This is in general not the case. Consider a function $u:\mathbb{R}^2\to\mathbb{R}$ of the form

$$u(x,y) = 1_{0<x<1}|x|^{ma}\psi(|x|^a y),$$

where m,a are real parameters, $\psi\in C_c^\infty(\mathbb{R})$. Then for $p,q\in(0,\infty)$ we have

$$\begin{aligned}\|u\|_{L_{xy}^p} &= \|\psi\|_{L^p}(\int_0^1|x|^{ap(m-1/p)}\,dx)^{1/p}\\ \|u\|_{L_{xy}^q} &= \|\psi\|_{L^q}(\int_0^1|x|^{aq(m-1/q)}\,dx)^{1/q}\\ \|u\|_{L_x^pL_y^q} &= \|\psi\|_{L^q}(\int_0^1|x|^{ap(m-1/q)}\,dx)^{1/p}.\end{aligned}$$

Suppose $p > q$. Then if $0 < m < 1/q$, there exists $a > 0$ such that

$$ap(m - \frac{1}{q}) < -1 < \min(ap(m - \frac{1}{p}), aq(m - \frac{1}{q})),$$

which implies $u \in L^p_{xy} \cap L^q_{xy}$ but $\|u\|_{L^p_x L^q_y} = \infty$.

Acknowledgements

We thank Prof. Tetsu Mizumachi for fruitful discussions in the initial stage of this project. Tsai's research is supported in part by NSERC grant 261356-13.

References

[1] T. Cazenave. *Semilinear Schrödinger equations*. New York University – Courant Institute, New York, 2003.

[2] Raphaël Côte and Stefan Le Coz. High-speed excited multi-solitons in nonlinear Schrödinger equations. *J. Math. Pures Appl. (9)*, 96(2):135–166, 2011.

[3] Raphaël Côte, Yvan Martel, and Frank Merle. Construction of multi-soliton solutions for the L^2-supercritical gKdV and NLS equations. *Rev. Mat. Iberoam.*, 27(1):273–302, 2011.

[4] Spyridon Kamvissis. Focusing nonlinear Schrödinger equation with infinitely many solitons. *J. Math. Phys.*, 36(8):4175–4180, 1995.

[5] S. Le Coz, D. Li, and T.-P. Tsai. Fast-moving finite and infinite trains of solitons for nonlinear Schrödinger equations. *Proceedings of the Royal Society of Edinburgh, Section: A Mathematics*, to appear.

[6] S. Le Coz and T.-P. Tsai. Infinite soliton and kink-soliton trains for nonlinear Schrödinger equations. *Nonlinearity*, 27:2689–2709, 2014.

[7] S. Le Coz and T.-P. Tsai. Finite and infinite soliton and kink-soliton trains of nonlinear Schrödinger equations. 2015. To appear in the proceedings of ICCM VI (Taipei 2013).

[8] Yvan Martel and Frank Merle. Multi solitary waves for nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 23(6):849–864, 2006.

[9] Yvan Martel, Frank Merle, and Tai-Peng Tsai. Stability in H^1 of the sum of K solitary waves for some nonlinear Schrödinger equations. *Duke Math. J.*, 133(3):405–466, 2006.

[10] Frank Merle. Construction of solutions with exactly k blow-up points for the Schrödinger equation with critical nonlinearity. *Comm. Math. Phys.*, 129(2):223–240, 1990.

[11] Galina Perelman. Asymptotic stability of multi-soliton solutions for nonlinear Schrödinger equations. *Comm. Partial Differential Equations*, 29(7-8):1051–1095, 2004.

[12] I Rodnianski, W Schlag, and A Soffer. Asymptotic stability of n-soliton states of nonlinear Schrödinger equations. preprint arXiv:math/0309114.

[13] V. E. Zakharov and A. B. Shabat. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Šoviet Physics JETP*, 34(1):62–69, 1972.